Residue-class distribution and mean values of multiplicative functions

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Partially based on joint work with Paul Pollack

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But for multiplicative functions, this is **NOT** the correct notion to consider. (Recall: f is multiplicative if $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$ such that $\gcd(m, n) = 1$.)

Let $\varphi(n)$ denote Euler's totient; that is, $\varphi(n) = \#(\mathbb{Z}/n\mathbb{Z})^\times$.

Fact: For a fixed q, $\varphi(n) \equiv 0 \pmod q$ for "almost all" positive integers n:

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For multiplicative functions $f : \mathbb{N} \to \mathbb{Z}$, it makes sense to study their distribution in the multiplicative group U_q mod q. So now our sample space is $\{n : \gcd(f(n), q) = 1\}$.

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Consider $f : \mathbb{N} \to \mathbb{Z}$ and $q \in \mathbb{N}$. We say f is **weakly uniformly distributed** (or **weakly equidistributed** or **WUD**) **modulo** q if:

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Consequence of general criterion for "polynomially-defined" multiplicative functions.

Explicit numerical distributions of $\varphi(n)$ mod 5:

For $x \geq 1$ and $r \in \{1, 2, 3, 4\}$ let

$$\rho_r(x) := \frac{\#\{n \leq x : \varphi(n) \equiv r \pmod 5\}}{\#\{n \leq x : \gcd(\varphi(n), 5) = 1\}}$$

x	$\rho_1(x)$	$\rho_2(x)$	$\rho_3(x)$	$\rho_4(x)$
10^5	0.27165	0.28003	0.23993	0.20837
10^6	0.27157	0.27556	0.23979	0.21307
10^7	0.27073	0.27267	0.23999	0.21660
10^8	0.26998	0.27051	0.24032	0.21917
10^9	0.26924	0.26884	0.24063	0.22127

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What fails mod 3? The numbers $p - 1$, for $p \neq 3$ prime, either fail to be coprime to 3 or are "trapped" in the trivial subgroup of $(\mathbb{Z}/3\mathbb{Z})^\times$.

(Jump back to slide 31)

One can similarly define a family $f_1, \cdots, f_K : \mathbb{N} \to \mathbb{Z}$ to be **jointly weakly equidistributed** or **(jointly WUD)** modulo $q \in \mathbb{N}$ if:

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A consequence of this: Let $\sigma(n) = \sum_{d|n} d$, $\sigma_2(n) = \sum_{d|n} d^2$.

Theorem 2.

$(\varphi, \sigma, \sigma_2)$ are jointly WUD modulo any fixed q s.t. $P^-(q) > 23$.

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Model (Siegel-Walfisz Theorem). Fix $K_0 > 0$. The primes $\leq x$ are weakly equidistributed mod q, uniformly for $q \leq (\log x)^{K_0}$. That is,

$$\frac{\#\{p \leq x : p \equiv a \pmod{q}\}}{\frac{1}{\varphi(q)}\#\{p \leq x\}} \to 1$$

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In other words, For any given $\epsilon > 0$, there exists $X(\epsilon, K_0)$ **depending only on** ϵ **and** K_0 s.t. the above ratio lies between $1 - \epsilon$ and $1 + \epsilon$ for all $x > X(\epsilon, K_0)$, all $q \leq (\log x)^{K_0}$ and all coprime residues a mod q.

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Question (made precise). Can we establish analogues of Siegel-Walfisz with primes replaced by values of φ or $(\varphi, \sigma, \sigma_2)$?

Theorem 3 (Pollack, S. R., 2022).

Fix $K_0 > 0$. As $x \to \infty$,

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Shortcomings of this result:

- Several arguments are restricted to a single multiplicative function and cannot be generalized to families, so cannot uniformize Nakiewicz's 1982-criterion.
- Even for a single multiplicative function, we are not able to recover a uniform version of Narkiewicz's 1967-criterion as we need to impose several additional restrictions on q and F.

In recent work, these shortcomings have been addressed. The main results of today's talk are extensions of Narkiewicz's general criterion for families of "polynomially-defined" multiplicative functions that are essentially **optimal** in the range of q and arithmetic restrictions on q.

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Theorem 4 (S. R., 2023).

Fix $\epsilon \in (0,1)$. As $x \to \infty$, we have

$$\frac{\#\{n \le x : (\varphi, \sigma, \sigma_2)(n) \equiv (a_1, a_2, a_3) \pmod{q}\}}{\frac{1}{\varphi(q)^3}\#\{n \le x : \gcd(\varphi\sigma\sigma_2(n), q) = 1\}} \to 1,$$

uniformly in moduli $q \le (\log x)^{(1/2-\epsilon)\alpha(q)}$ having $P^-(q) > 23$ and in coprime residue classes a_i mod q, where

$$\alpha(q) = \frac{1}{\varphi(q)}\#\{u \in U_q : (u-1)(u+1)(u^2+1) \in U_q\}$$
$$= \prod_{\ell \mid q:\ \ell \equiv -1 \pmod{4}} \left(1 - \frac{2}{\ell - 1}\right) \cdot \prod_{\ell \mid q:\ \ell \equiv 1 \pmod{4}} \left(1 - \frac{4}{\ell - 1}\right).$$

Extending uniformity to the Siegel–Walfisz range:

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Work-around: Restrict to inputs n having sufficiently many large prime factors. Equidistribution is restored among these inputs.

Theorem 5 (S. R., 2023).

Fix $K_0 > 0$ *and* $\epsilon \in (0,1)$. *We have*

$$\#\{n \le x : P_{13}(n) > q, (\varphi, \sigma, \sigma_2)(n) \equiv (a_1, a_2, a_3) \pmod q\} \sim \frac{1}{\varphi(q)^3} \#\{n \le x : P_{13}(n) > q, \gcd(\varphi\sigma\sigma_2(n), q) = 1\},$$

as $x \to \infty$, *uniformly in* $q \le (\log x)^{K_0}$ *satisfying* $P^-(q) > 23$ *and in coprime residues* a_i *mod* q.

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For squarefree q, *"13" can be replaced by "7".*

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This mixing plays a central role for WUD of $\varphi(n)$.

Some central themes behind the arguments:

1. Exploit a "mixing" phenomenon in U_q (quantitative ergodicity phenomenon for random walks in U_q).

Heuristic: Assume $\gcd(q, 6) = 1$ and let $\mathcal{R}' = \{u \in U_q : u - 1 \in U_q\}$. Choose uniformly at random $u_1, u_2, u_3, \ldots$ from $\mathcal{R}'$, and consider the products

$$u_1 - 1, (u_1 - 1)(u_2 - 1), (u_1 - 1)(u_2 - 1)(u_3 - 1), \ldots$$

Mixing in U_q: As $J \to \infty$, each element of U_q becomes roughly equally likely to appear as one of the products $\prod_{j=1}^{J}(u_j - 1)$.

This mixing plays a central role for WUD of $\varphi(n)$. In the case of $(\varphi, \sigma, \sigma_2)$, the analogous mixing phenomenon is that of the tuples $(u - 1, u + 1, u^2 + 1)$ in the group U_q^3, where $u_1, u_2, u_3, \ldots$ are chosen from the set $\mathcal{R} = \{u \in U_q : (u - 1)(u + 1)(u^2 + 1) \in U_q\}$.

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- Need to consider certain regular sequences in $\overline{\mathbb{F}}_\ell[X_1, \ldots, X_r]$.

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Convenient $n \le x$ give dominant contribution: After some careful "anatomical" arguments, we can reduce proving Theorems 4 and 5 to showing that

Theorem 6 (Workhorse Result).

Let $f = \varphi\sigma\sigma_2$. As $x \to \infty$, we have

$$\#\{n \le x \ \ \mathrm{conv} : (\varphi, \sigma, \sigma_2)(n) \equiv (a_1, a_2, a_3) \pmod{q}\} \\ \sim \frac{1}{\varphi(q)^3} \#\{n \le x : \gcd(f(n), q) = 1\},$$

uniformly in $q \le (\log x)^{K_0}$ s.t. $P^-(q) > 23$ and uniformly in $a_i \in U_q$.

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First step: Reduction to bounded divisor

Proposition 1.

In the above setting, there exists $Q_0 \mid q$ s.t. $Q_0 = O(1)$ and

$$\#\{n \le x \ \text{conv} : (\varphi, \sigma, \sigma_2)(n) \equiv (a_1, a_2, a_3) \pmod{q}\} \approx \frac{1}{\varphi(q)^3} \cdot \varphi(Q_0)^3 \#\{n \le x : (f(n), q) = 1, (\varphi, \sigma, \sigma_2)(n) \equiv (a_1, a_2, a_3) \pmod{Q_0}\}$$

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Any convenient n can be written as $mP_J \dots P_1$ where $\max\{y, P(m)\} < P_J < \cdots < P_1$. Then $\varphi(n) = \varphi(m) \prod_{j=1}^{J} (P_j - 1)$.

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Thus

$$\varphi(n) \equiv a_1,\ \sigma(n) \equiv a_2,\ \sigma_2(n) \equiv a_3 \bmod q$$
$$\iff (P_1, \dots, P_J) \equiv (v_1, \dots, v_J) \bmod q$$

for some $(v_1, \dots, v_J) \in U_q^J$ satisfying:
(i) $\prod_{j=1}^{J}(v_j - 1) \equiv a_1\varphi(m)^{-1}$, (ii) $\prod_{j=1}^{J}(v_j + 1) \equiv a_2\sigma(m)^{-1}$, (iii) $\prod_{j=1}^{J}(v_j^2 + 1) \equiv a_3\sigma_2(m)^{-1} \pmod{q}$.

Let $V_{q,m}$ denote the set of such $(v_1, \dots, v_J)$.

By J careful applications of Siegel–Walfisz,

$$\sum_{\substack{n \le x \text{ conv} \\ (\varphi,\sigma,\sigma_2)(n) \equiv (a_1,a_2,a_3) \bmod q}} 1 \approx \sum_{\substack{m \le x \\ \text{blah}}} \frac{\# V_{q,m}}{\varphi(q)^J} \sum_{\substack{P_1,\ldots,P_J \\ \text{more blah}}} 1$$

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Fact 1: $\exists Q_0 \mid q$ s.t. $Q_0 = O(1)$ and uniformly in m,

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A less standard key ingredient: Linear algebra over rings.

Note: Here, it is crucial that the three polynomials $T-1$, $T+1$ and T^2+1 are "multiplicatively independent" over $\mathbb{Z}$, i.e, for any integers $(c_1, c_2, c_3) \neq (0,0,0)$, we have $(T-1)^{c_1}(T+1)^{c_2}(T^2+1)^{c_3} \neq$ constant.

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Combining,

$$\sum_{\substack{n\le x \text{ conv} \\ (\varphi,\sigma,\sigma_2)(n)\equiv(a_1,a_2,a_3) \bmod q}} 1$$

$$\approx \left(\frac{\varphi(Q_0)}{\varphi(q)}\right)^3 \cdot \left(\frac{\alpha(q)}{\alpha(Q_0)}\right)^J \sum_{\substack{m\le x \\ \text{blah}}} \frac{\#V_{Q_0,m}}{\varphi(Q_0)^J} \sum_{\substack{P_1,\dots,P_J \\ \text{more blah}}} 1.$$

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After some more technical arguments,

$$\sum_{\substack{n\le x \text{ conv} \\ (\varphi,\sigma,\sigma_2)(n)\equiv(a_1,a_2,a_3) \bmod q}} 1 \approx \left(\frac{\varphi(Q_0)}{\varphi(q)}\right)^3 \sum_{\substack{n\le x:\ (f(n),q)=1 \\ (\varphi,\sigma,\sigma_2)(n)\equiv(a_1,a_2,a_3) \bmod Q_0}} 1.$$

This completes our initial reduction step (to bounded modulus Q_0).

The analytic argument I

We have shown: $\exists\ Q_0 \mid q$ s.t. $Q_0 = O(1)$, and

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Now apply orthogonality to detect congruences mod Q_0. Enough to show:

Proposition 2.

For any $\widehat{\chi} = (\chi_1, \chi_2, \chi_3) \neq (\chi_0, \chi_0, \chi_0)$ mod Q_0, the sum

$$\sum_{n \le x} \mathbb{1}_{(f(n), q) = 1} \cdot \chi_1(\varphi(n)) \chi_2(\sigma(n)) \chi_3(\sigma_2(n))$$

is negligible compared to $\#\{n \le x : \gcd(f(n), q) = 1\}$.

Case 1: When $\mathbb{1}_{(u,Q_0)=1} \cdot \chi_1(u-1)\chi_2(u+1)\chi_3(u^2+1)$ is **not** constant on its support.

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Key tool:

Theorem 7 (Halász).

Let F be a multiplicative function s.t. $|F(n)| \le 1$ for all n. For $x, T \ge 2$,

$$\frac{1}{x}\sum_{n\le x} F(n) \ll \frac{1}{T} + \exp\left(-\min_{|t|\le T}\sum_{p\le x}\frac{1-\mathrm{Re}(F(p)p^{-it})}{p}\right).$$

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Apply this to $F(n) = \mathbb{1}_{(f(n),q)=1} \cdot \chi_1(\varphi(n))\chi_2(\sigma(n))\chi_3(\sigma_2(n))$.

Obtaining a lower bound on

$$\sum\nolimits^*_{p \le x} \frac{1}{p} \cdot \left(1 - \mathrm{Re}(p^{-it}\chi_1(p-1)\chi_2(p+1)\chi_3(p^2+1))\right).$$

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Remark: For the resulting lower bound to be nontrivial, we need our hypothesis that $\mathbb{1}_{(u,Q_0)=1}\cdot\chi_1(u-1)\chi_2(u+1)\chi_3(u^2+1)$ is **not** constant on its support.

Case 2: When $\mathbb{1}_{(u,Q_0)=1} \cdot \chi_1(u-1)\chi_2(u+1)\chi_3(u^2+1)$ is constant on its support.

Recall: Want to show that

$$\sum_{n \le x} \mathbb{1}_{(f(n),q)=1} \cdot \chi_1(\varphi(n))\chi_2(\sigma(n))\chi_3(\sigma_2(n))$$

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Note: Possible essential singularity at $s = 1$.

The modification

We identify our sum

$$\sum_{n\leq x} \mathbb{1}_{(f(n),q)=1} \cdot \chi_1(\varphi(n))\chi_2(\sigma(n))\chi_3(\sigma_2(n))$$

as the partial sum of the Dirichlet series

$$F_{\widehat{\chi}}(s) = \sum_{n=1}^{\infty} \frac{\mathbb{1}_{(f(n),q)=1}}{n^s} \cdot \chi_1(\varphi(n))\chi_2(\sigma(n))\chi_3(\sigma_2(n)).$$

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But here

$$F_{\widehat{\chi}}(s) \approx \left(\prod_{\substack{d\mid q \\ d \text{ sqfree}}} \prod_{\substack{\psi \bmod d \\ \psi \text{ primitive}}} L(s,\psi)^{\gamma(\psi)} \right)^{\alpha(q)c_{\widehat{\chi}}}$$

Here $c_{\widehat{\chi}} = \mathbb{1}_{(u,Q_0)=1} \cdot \chi_1(u-1)\chi_2(u+1)\chi_3(u^2+1) \neq 0$.

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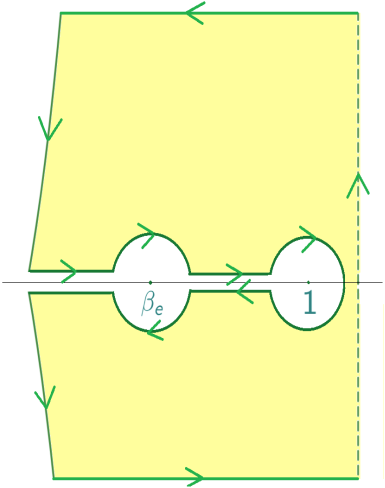
So we modify the usual "LSD contour" into the adjacent one.



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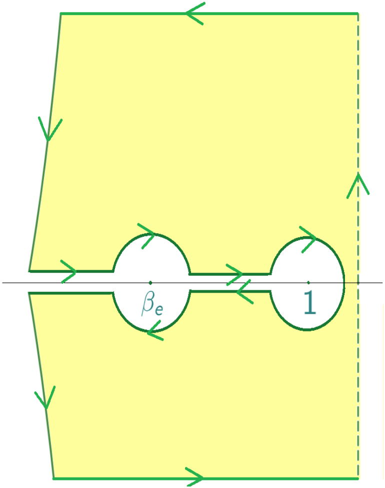

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Technicalities: almost entirely different from usual LSD (partly inspired from work of Scourfield).

After a lot of technical work, we deduce that if $P^{-}(q) > 23$, then

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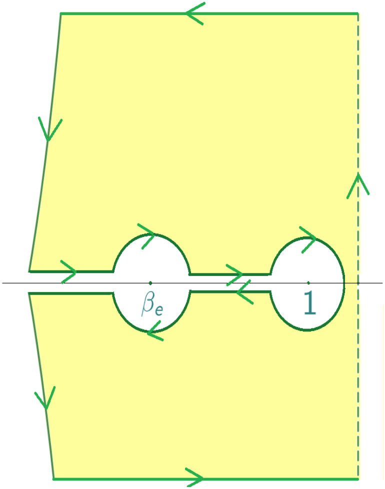
This completes the proof of our Workhorse result Theorem 6, and hence also of Theorems 4 and 5.



(Some of) the General Main Results

Consider multiplicative functions $f_1, \ldots, f_K : \mathbb{N} \to \mathbb{Z}$ and polynomials $\{W_{i,v}\}_{\substack{1 \le i \le K \\ 1 \le v \le V}} \subset \mathbb{Z}[T]$, such that $f_i(p^v) = W_{i,v}(p)$.

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Given $k \in \{1, \ldots, V\}$, we say that q is **k-admissible** if
$\{u \in U_q : (\forall i)\ W_{i,k}(u) \in U_q\} \neq \emptyset$, but
$\{u \in U_q : (\forall i)\ W_{i,v}(u) \in U_q\} = \emptyset$, for each $1 \le v \le k-1$.

Notation: For a fixed $k \in \{1, \ldots, V\}$, define

$\mathcal{Q}(k; f_1, \cdots, f_K) := \{q : \ q \text{ is } k\text{-admissible, } f_1, \ldots, f_K \text{ are jointly WUD mod } q\}$.

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We give uniform analogues of Narkiewicz's result, which are best possible in the range and arithmetic restrictions on q. We just need two technical hypotheses H_1 and H_2, which we can prove to be necessary.

Let $\alpha_k(q) = \frac{1}{\varphi(q)} \#\{u \in U_q : \prod_{i=1}^{K} W_{i,k}(u) \in U_q\}$ and $D_{\min} = \min_{1 \le i \le K} \deg(W_{i,k})$.

Theorem 8 (S.R., 2023).

Fix $K_0 > 0$ and $\epsilon \in (0,1)$. Under H_1 and H_2, the functions $f_1, \dots, f_K$ are jointly WUD, uniformly modulo $q \in \mathcal{Q}(k; f_1, \cdots, f_K)$, provided any **one** *of the following holds.*

(i) $$q \le \begin{cases} (\log x)^{K_0}, & \text{if } K = 1 \text{ and } W_{1,k} \text{ is linear.} \\ (\log x)^{(1-\epsilon)\alpha_k(q)(K - 1/D_{min})^{-1}}, & \text{otherwise.} \end{cases}.$$

(ii) *q is squarefree and $q^{K-1} D_{min}^{\omega(q)} \le (\log x)^{(1-\epsilon)\alpha_k(q)}$.*

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Optimality: This result is essentially optimal in the arithmetic restrictions on q as well as in the hypotheses H_1 and H_2. Also, second case of (i) and (ii) are optimal in the range of q.

As for $\varphi, \sigma, \sigma_2$, we need to restrict our input sets to get complete uniformity up to arbitrary powers of $\log x$. Fix $K_0 > 0$.

Theorem 9 (S.R., 2023).

Fix $K_0 > 0$. Under H_1 and H_2, we have

$$\#\{n \le x : P_R(n) > q, \ \ (\forall i)\ f_i(n) \equiv a_i \pmod q\} \sim \frac{1}{\varphi(q)^K} \# \left\{ n \le x : P_R(n) > q,\ \gcd(\prod_{i=1}^{K} f_i(n), q) = 1 \right\},$$

uniformly in $q \le (\log x)^{K_0}$ lying in $\mathcal{Q}(k; f_1, \cdots, f_K)$ and in $a_1, \ldots, a_K \in U_q$. Here

1. $R = \max\{k(KD+1), k(1 + (k+1)(K - 1/D))\}$ *for general q.*
2. *If q is squarefree and $k \ge 2$, then*

$$R = \begin{cases} k(Kk + K - k) + 1, & \text{if one of } \{W_{i,k}\}_{i=1}^{K} \text{ not sqfull.} \\ k(Kk + K - k + 1) + 1, & \text{in general.} \end{cases}$$

3. *If q is squarefree and $k = 1$, then $R = 2K + 1$.*
   Further, if $k = K = 1$ and $W_{1,k}$ is not squarefull, then $R = 2$.

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Optimality: Most of these R's are either exactly or nearly optimal, ensuring joint WUD among as large a set of inputs as possible.

Ongoing work: Finer distribution questions and an extension of the Landau–Selberg–Delange method

Question: Can we say anything about the deviation of $\#\{n \le x : (\forall i)\ f_i(n) \equiv a_i \pmod{q}\}$ from its expected value $\frac{1}{\varphi(q)^K} \#\{n \le x, (\forall i)\ \gcd(f_i(n), q) = 1\}$, uniformly for $q \le (\log x)^{K_0}$?

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To say something interesting, we will need precise asymptotics for the sums $\sum_{n \le x} \chi_1(f_1(n)) \ldots \chi_K(f_K(n))$ in the full range $q \le (\log x)^{K_0}$.

General question (extension of LSD): Assuming that $\sum_{n\geq 1} a_n/n^s \approx \prod_{\chi \bmod q} L(sk,\chi)^{\alpha_\chi}$, give **precise** asymptotic series estimating $\sum_{n\leq x} a_n$ **uniformly** in q in a wide range.

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Theorem 10 (S.R. 2024, in preparation).

Fix $K_0 > 0$. In the above setting and under some natural additional hypotheses, we have

$$\sum_{n \leq x} a_n = \frac{x^{1/k}}{(\log x)^{1-\alpha_{\chi_0}}} \sum_{0 \leq j \leq N} \frac{\mu_j}{(\log x)^j} \; + \; O(\text{error term}),$$

uniformly in $x \geq 3$, $N \geq 0$ and $q \leq (\log x)^{K_0}$. The error term is genuinely smaller than the main term in the full range $q \leq (\log x)^{K_0}$.

Concrete and potential applications (ongoing work):

1. Estimate $\#\{n \le x : \gcd(f(n), q) = 1\}$ for large classes of multiplicative functions f.

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- Theorem 10: precise estimates for larger classes of f, **uniformly** in $q \le (\log x)^{K_0}$.
- Extra generality with "k" allows us to consider more interesting varieties of f and q, for which behavior of f at higher prime powers becomes crucial. (Eg.: $\sigma(n)$ for $2 \mid q$: Behavior at p^2 matters.)

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(1) Positive integers with prime divisors restricted to residue classes:
Given $q \in \mathbb{N}$ and $\mathcal{A} \subset U_q$, estimate
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(2) Distributions of the least invariant factor of multiplicative groups: Writing $U_n = \mathbb{Z}/\lambda_1\mathbb{Z} \oplus \mathbb{Z}/\lambda_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\lambda_r\mathbb{Z}$ with $\lambda_1 \mid \lambda_2 \mid \cdots \mid \lambda_r$, let $\lambda_1(n) := \lambda_1$. Estimate $\#\{n \le x : \lambda_1(n) = d\}$.

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- Chang–Martin (2020): Do this for fixed d.
- Theorem 10: Uniformly in $d \le (\log x)^{K_0}$ with much better error terms.

Thank you for your attention!

Email: akash01s.roy@gmail.com