Joint distribution in residue classes of families of "polynomially-defined" multiplicative functions

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Partially based on joint work with Paul Pollack

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Let f be an integer-valued arithmetic function and q be a positive integer. We say f is **uniformly distributed** (or **equidistributed**) **modulo** q if, for each integer a,

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But for multiplicative functions, this is **NOT** the correct notion to consider. (Recall: f is multiplicative if $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{Z}^+$ such that $\gcd(m, n) = 1$.)

Let $\varphi(n)$ denote Euler's totient; that is, $\varphi(n) = \#(\mathbb{Z}/n\mathbb{Z})^\times$.

Fact: For a fixed q, $\varphi(n) \equiv 0 \pmod{q}$ for "almost all" positive integers n:

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For multiplicative functions $f : \mathbb{N} \to \mathbb{Z}$, it makes sense to study their distribution in the multiplicative group U_q mod q. So now our sample space is $\{n : \gcd(f(n), q) = 1\}$.

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2. for each $a \in U_q$,

$$\frac{\#\{n \le x : f(n) \equiv a \pmod q\}}{\frac{1}{\varphi(q)}\#\{n \le x : \gcd(f(n), q) = 1\}} \to 1,$$

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Consequence of general criterion for "polynomially-defined" multiplicative functions.

One can similarly define a family $f_1, \cdots, f_K$ of arithmetic functions to be **jointly weakly equidistributed** or **(jointly WUD)** modulo q if:

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A consequence of this: Let $\sigma(n) = \sum_{d|n} d$, $\sigma_2(n) = \sum_{d|n} d^2$.

Theorem 2.
$(\varphi, \sigma, \sigma_2)$ are jointly WUD modulo any fixed q s.t. $P^-(q) > 23$.

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Model (Siegel-Walfisz Theorem). Fix $K_0 > 0$. The primes $\leq x$ are weakly equidistributed mod q, uniformly for $q \leq (\log x)^{K_0}$. That is,

$$\frac{\#\{p \leq x : p \equiv a \pmod q\}}{\frac{1}{\varphi(q)}\#\{p \leq x\}} \to 1$$

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In other words, For any given $\epsilon > 0$, there exists $X(\epsilon, K_0)$ **depending only on** ϵ **and** K_0 s.t. the above ratio lies between $1-\epsilon$ and $1+\epsilon$ for all $x > X(\epsilon)$, all $q \leq (\log x)^{K_0}$ and all coprime residues a mod q.

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Question (made precise). Can we establish analogues of Siegel-Walfisz with primes replaced by values of φ or $(\varphi, \sigma, \sigma_2)$?

Theorem 3 (Pollack, S. R., 2022).

Fix $K_0 > 0$. As $x \to \infty$,

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Shortcomings of this result:

- Several arguments are restricted to a single multiplicative function and cannot be generalized to families.
- Even for a single multiplicative function, we are not able to recover a uniform version of Narkiewicz's general criterion as we need to impose several additional restrictions on q and F.

In recent work, these shortcomings have been addressed. The main results of today's talk are extensions of Narkiewicz's general criterion for families of "polynomially-defined" multiplicative functions that are **optimal** in the range and arithmetic restrictions of q as well as all almost all other hypotheses.

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Theorem 4 (S. R., 2023).

Fix $\epsilon \in (0,1)$. As $x \to \infty$, we have

$$\frac{\#\{n \le x : \varphi(n) \equiv a_1, \sigma(n) \equiv a_2, \sigma_2(n) \equiv a_3 \pmod q\}}{\frac{1}{\varphi(q)^3}\#\{n \le x : \gcd(\varphi\sigma\sigma_2(n), q) = 1\}} \to 1,$$

uniformly in moduli $q \le (\log x)^{(1/2-\epsilon)\alpha(q)}$ having $P^-(q) > 23$ and in coprime residue classes a_i mod q, where

$$\alpha(q) = \frac{1}{\varphi(q)}\#\{u \in U_q : (u-1)(u+1)(u^2+u+1) \in U_q\}$$
$$= \prod_{\ell | q:\ \ell \not\equiv 1 \pmod 3}\left(1 - \frac{2}{\ell-1}\right) \cdot \prod_{\ell | q:\ \ell \equiv 1 \pmod 3}\left(1 - \frac{4}{\ell-1}\right).$$

Remark: $\varphi(P) = P-1$, $\sigma(P) = P+1$, $\sigma_2(P) = P^2+P+1$.

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Work-around: Restrict to inputs n having sufficiently many large prime factors. Equidistribution is restored among these inputs.

Theorem 5 (S. R., 2023).

Fix $K_0 > 0$ *and* $\epsilon \in (0,1)$. *We have*

$$\#\{n \le x : P_{13}(n) > q, \varphi(n) \equiv a_1, \sigma(n) \equiv a_2, \sigma_2(n) \equiv a_3 \pmod q\}$$
$$\sim \frac{1}{\varphi(q)^3} \#\{n \le x : P_{13}(n) > q, \gcd(\varphi\sigma\sigma_2(n), q) = 1\},$$

as $x \to \infty$, *uniformly in* $q \le (\log x)^{K_0}$ *satisfying* $P^-(q) > 23$ *and in coprime residues* a_i *mod* q.

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as $x \to \infty$, uniformly in $q \le (\log x)^{K_0}$ satisfying $P^-(q) > 23$ and in coprime residues a_i mod q.

For squarefree q, "13" can be replaced by "7".

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This mixing plays a central role for WUD of $\varphi(n)$. In our case, the analogous mixing phenomenon is that of the tuples $(u-1, u+1, u^2+u+1)$ in the group U_q^3, where $u_1, u_2, u_3, \dots$ are chosen from the set $\mathcal{R} = \{u \in U_q : (u-1)(u+1)(u^2+u+1) \in U_q\}$.

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- Need to consider certain regular sequences in $\overline{\mathbb{F}}_\ell[X_1, \ldots, X_r]$.

A crude estimate for the main term:

Let $f = \varphi\sigma\sigma_2$.

Recall that

$$\alpha(q) = \frac{1}{\varphi(q)} \#\{u \in U_q : (u-1)(u+1)(u^2+u+1) \in U_q\}.$$

Proposition 1.
Uniformly in $q \le (\log x)^{K_0}$ s.t. $P^-(q) > 7$, we have,

$$\#\{n \le x : \gcd(f(n), q) = 1\} \asymp \frac{x}{(\log x)^{1-\alpha(q)}} \cdot (\text{negligible factors})$$

The major contributors: Convenient n

Let $J = J(x)$ be an integer going to infinity very slowly, say

$$J = \lfloor \log \log \log x \rfloor.$$

Let

$$y = \exp((\log x)^{\epsilon/2})$$

(ϵ as in statement of Thm 4, $\epsilon = 1$ for Thm 5).
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Note $q \ll y \ll x^{1/1000}$.

Convenient n: The largest J prime factors of n are $> y$ and distinct.
In other words, $n = mP_J \dots P_1$, where
$\max\{y, P(m)\} < P_J < \cdots < P_1$.

The major contributors: Convenient n

Let $J = J(x)$ be an integer going to infinity very slowly, say

$$J = \lfloor \log\log\log x \rfloor.$$

Let

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(ϵ as in statement of Thm 4, $\epsilon = 1$ for Thm 5).
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Reason y? Past y, primes are very regularly distributed in coprime residue classes mod q, when $q \le (\log x)^{K_0}$.

Convenient n give dominant contribution.

Theorem 6 (Workhorse Result).

As $x \to \infty$, we have

$$\#\{n \le x \ \text{conv} : \varphi(n) \equiv a_1, \sigma(n) \equiv a_2, \sigma_2(n) \equiv a_3 \pmod q\} \\ \sim \frac{1}{\varphi(q)^3} \#\{n \le x : \gcd(f(n), q) = 1\},$$

uniformly in $q \le (\log x)^{K_0}$ s.t. $P^-(q) > 23$ and uniformly in $a_i \in U_q$.

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First step: Reduction to bounded divisor

Proposition 2.

In the above setting, there exists $Q_0 \mid q$ s.t. $Q_0 = O(1)$ and

$$\#\{n \le x \ \text{conv} : \varphi(n) \equiv a_1, \sigma(n) \equiv a_2, \sigma_2(n) \equiv a_3 \pmod q\} \approx \frac{1}{\varphi(q)^3} \cdot \varphi(Q_0)^3 \#\{n \le x : (f(n), q) = 1, \varphi(n) \equiv a_1, \sigma(n) \equiv a_2, \sigma_2(n) \equiv a_3 \pmod{Q_0}\}$$

The first step: Reduction to bounded modulus.

Any convenient n can be written as $mP_J \dots P_1$ where $\max\{y, P(m)\} < P_J < \cdots < P_1$. Then $\varphi(n) = \varphi(m) \prod_{j=1}^{J} (P_j - 1)$.

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Thus

$$\varphi(n) \equiv a_1,\ \sigma(n) \equiv a_2,\ \sigma_2(n) \equiv a_3 \bmod q \iff (P_1, \dots, P_J) \bmod q \in V_{q,m}$$

where $V_{q,m}$ denotes the set of such $(v_1, \dots, v_J) \in U_q^J$ that satisfy:
(i) $\prod_{j=1}^{J}(v_j - 1) \equiv a_1 \varphi(m)^{-1}$,
(ii) $\prod_{j=1}^{J}(v_j + 1) \equiv a_2 \sigma(m)^{-1}$,
(iii) $\prod_{j=1}^{J}(v_j^2 + v_j + 1) \equiv a_3 \sigma_2(m)^{-1} \pmod{q}$.

Thus

$$\sum_{\substack{n\le x \text{ conv} \\ \varphi(n)\equiv a_1,\ \sigma(n)\equiv a_2 \\ \sigma_2(n)\equiv a_3 \pmod q}} 1 = \sum_{\substack{m\le x \\ \text{blah}}} \sum_{\substack{P_1,\dots,P_J \\ \text{more blah} \\ (P_1,\dots,P_J) \bmod q \in V_{q,m}}} 1.$$

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By J careful applications of Siegel–Walfisz,

$$\sum_{\substack{n\le x \text{ conv} \\ \varphi(n)\equiv a_1,\ \sigma(n)\equiv a_2 \\ \sigma_2(n)\equiv a_3 \pmod q}} 1 \approx \sum_{\substack{m\le x \\ \text{blah}}} \frac{\#V_{q,m}}{\varphi(q)^J} \sum_{\substack{P_1,\dots,P_J \\ \text{more blah}}} 1$$

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Fact 1: $\exists Q_0 \mid q$ s.t. $Q_0 = O(1)$ and

$$\frac{\# V_{q,m}}{\varphi(q)^J} \approx \left(\frac{\varphi(Q_0)}{\varphi(q)}\right)^3 \cdot \left(\frac{\alpha(q)}{\alpha(Q_0)}\right)^J \frac{\# V_{Q_0,m}}{\varphi(Q_0)^J}.$$

Combining,

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After some more technical arguments, we get our initial reduction step:

$$\sum_{\substack{n\le x \text{ conv} \\ \varphi(n)\equiv a_1,\ \sigma(n)\equiv a_2 \\ \sigma_2(n)\equiv a_3 \pmod q}} 1 \approx \left(\frac{\varphi(Q_0)}{\varphi(q)}\right)^3 \sum_{\substack{n\le x:\ (f(n),q)=1 \\ \varphi(n)\equiv a_1,\ \sigma(n)\equiv a_2 \\ \sigma_2(n)\equiv a_3 \pmod{Q_0}}} 1.$$

More on Fact 1

Given $N \geq 1$ and $\mathbf{w} = (w_i)_{i=1}^3 \in U_q^3$, let

$$V_N(q, \mathbf{w}) = \{(v_1, \ldots, v_N) \in U_q^N : \prod_{j=1}^N (v_j - 1) \equiv w_1, \prod_{j=1}^N (v_j + 1) \equiv w_2, \prod_{j=1}^N (v_j^2 + v_j + 1) \equiv w_3 \pmod q\},$$

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so that $V_{q,m} = V_J(q, (a_1\varphi(m)^{-1}, a_2\sigma(m)^{-1}, a_3\sigma_2(m)^{-1}))$.

Fact 1 (Generalized): Consider q having $P^-(q) > 23$. Then $\exists Q_0 | q$, s.t. $Q_0 = O(1)$, and s.t. for $\mathbf{w} = (w_i)_{i=1}^3 \in U_q^3$ and $N \geq 13$,

$$\frac{\# V_N(q, \mathbf{w})}{(\alpha(q)\varphi(q))^N} \approx \left(\frac{\varphi(Q_0)}{\varphi(q)}\right)^3 \cdot \frac{\# V_N(Q_0, \mathbf{w})}{(\alpha(Q_0)\varphi(Q_0))^N}.$$

Recall $\alpha(q) = \frac{1}{\varphi(q)} \#\{u \in U_q : (u-1)(u+1)(u^2+u+1) \in U_q\}$.

Instead of $V_N(q, \mathbf{w})$, we consider, for $\ell^e \parallel q$,

$$V_N(\ell^e, \mathbf{w}) = \{(v_1, \ldots, v_N) \in U_{\ell^e}^N : \prod_{j=1}^{N}(v_j - 1) \equiv w_1, \prod_{j=1}^{N}(v_j + 1) \equiv w_2, \prod_{j=1}^{N}(v_j^2 + v_j + 1) \equiv w_3 \pmod{\ell^e}\},$$

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By the orthogonality of Dirichlet characters,

$$\# V_N(\ell^e, \mathbf{w}) = \frac{1}{\varphi(\ell^e)^3} \sum_{\chi_1, \chi_2, \chi_3 \bmod \ell^e} \overline{\chi}_1(w_1) \overline{\chi}_2(w_2) \overline{\chi}_3(w_3) (Z_{\ell^e, \widehat{\chi}})^N :$$

$Z_{\ell^e, \widehat{\chi}} = \sum_{v \in U_{\ell^e}} \chi_1(v - 1)\chi_2(v + 1)\chi_3(v^2 + v + 1)$ for $\widehat{\chi} = (\chi_1, \chi_2, \chi_3) \bmod \ell^e$.

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Character sum machinery allows us to show that the contribution of all the tuples $Z_{\ell^e, \widehat{\chi}}$ is negligible, for $\widehat{\chi} \neq (\chi_0, \chi_0, \chi_0) \bmod \ell^e$.

Note: Here, it is crucial that the three polynomials $T-1$, $T+1$ and T^2+T+1 are "multiplicatively independent" over $\mathbb{Q}$, i.e, for any $(c_1, c_2, c_3) \neq (0,0,0)$, we have
$(T-1)^{c_1}(T+1)^{c_2}(T^2+T+1)^{c_3} \neq$ constant.

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We get uniformly in $N \geq 13$ and in $\ell^e \parallel q$ for suff large ℓ,

$$\frac{\# V_N(\ell^e, \mathbf{w})}{(\alpha(\ell^e)\varphi(\ell^e))^N} \approx \frac{1}{\varphi(\ell^e)^3}$$

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This gives Fact 1, and completes the reduction to a bounded divisor.

The analytic argument I

Thus: $\exists\ Q_0 \mid q$ s.t. $Q_0 = O(1)$, and

$$\#\{n \le x \ \ \mathrm{conv} : \varphi(n) \equiv a_1, \sigma(n) \equiv a_2, \sigma_2(n) \equiv a_3 \pmod{q}\}$$
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Now apply orthogonality on the right hand side! Enough to show:

Proposition 3.

$\exists \delta_0 > 0$ s.t. for any $\widehat{\chi} = (\chi_1, \chi_2, \chi_3) \ne (\chi_0, \chi_0, \chi_0) \bmod Q_0$,

$$\sum_{n \le x} \mathbb{1}_{(f(n), q) = 1} \cdot \chi_1(\varphi(n))\chi_2(\sigma(n))\chi_3(\sigma_2(n)) \ll \frac{x}{(\log x)^{1-(1-\delta_0)\alpha(q)}}.$$

We first show this in the case when the product $\mathbb{1}_{(u,Q_0)=1} \cdot \chi_1(u-1)\chi_2(u+1)\chi_3(u^2+u+1)$ is **not** constant on its support.

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Key tool:

Theorem 7 (Halász).

Let F be a multiplicative function s.t. $|F(n)| \leq 1$ for all n. For $x, T \geq 2$,

$$\frac{1}{x}\sum_{n\leq x} F(n) \ll \exp\left(-\min_{|t|\leq T}\sum_{p\leq x}\frac{1-\mathrm{Re}(F(p)p^{-it})}{p}\right),$$

up to other negligible terms.

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up to other negligible terms.

For this bound to be useful, we need to **lower bound** the sums

$$\sum\nolimits^*_{p\leq x}\frac{1}{p}\cdot\left(1-\mathrm{Re}(p^{-it}\chi_1(p-1)\chi_2(p+1)\chi_3(p^2+p+1))\right).$$

Obtaining a lower bound on

$$\sum_{p\le x}^{*} \frac{1}{p} \cdot \left(1 - \mathrm{Re}(p^{-it}\chi_1(p-1)\chi_2(p+1)\chi_3(p^2+p+1))\right).$$

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Remark: For the resulting lower bound to be nontrivial, we need our hypothesis that $\mathbb{1}_{(u,Q_0)=1} \cdot \chi_1(u-1)\chi_2(u+1)\chi_3(u^2+u+1)$ is **not** constant on its support.

The analytic argument II

Want to show: $\exists \delta_0 > 0$ s.t. for any $\widehat{\chi} = (\chi_1, \chi_2, \chi_3) \neq (\chi_0, \chi_0, \chi_0)$ mod Q_0 for which $\mathbb{1}_{(u,Q_0)=1} \cdot \chi_1(u-1)\chi_2(u+1)\chi_3(u^2+u+1)$ is constant on its support, we have

$$\sum_{n \leq x} \mathbb{1}_{(f(n),q)=1} \cdot \chi_1(\varphi(n))\chi_2(\sigma(n))\chi_3(\sigma_2(n)) \ll \frac{x}{(\log x)^{1-(1-\delta_0)\alpha(q)}}.$$

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Key idea: We modify the Landau–Selberg–Delange (LSD) method.

The standard LSD method (Tenenbaum):

Given: Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \zeta(s)^z \cdot H(s)$$

where $z \in \mathbb{C}$ and $H(s)$ is very well-behaved.

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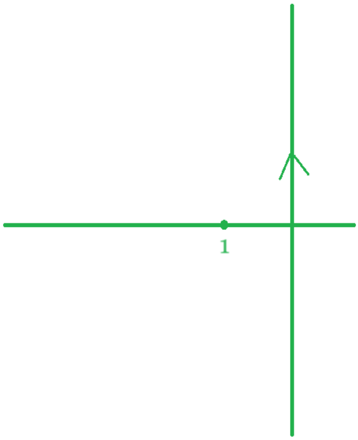
Note: Possible essential singularity at $s = 1$.

1. Perron's formula: Write $\sum_{n\leq x} a_n$ in terms of a complex integral involving $\sum_{n=1}^{\infty} a_n/n^s$, over a truncated vertical line to the right of 1.



1. Write $\sum_{n \leq x} a_n$ in terms of a complex integral involving $\sum_{n=1}^{\infty} a_n/n^s$, over a truncated vertical line to the right of 1.

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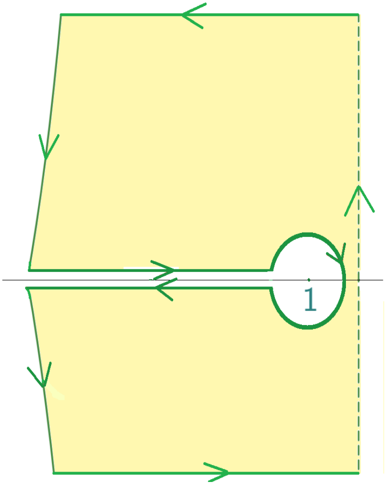

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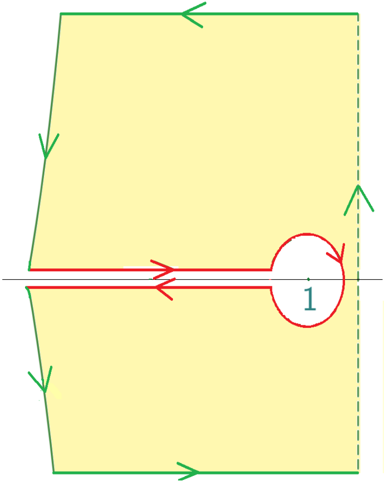

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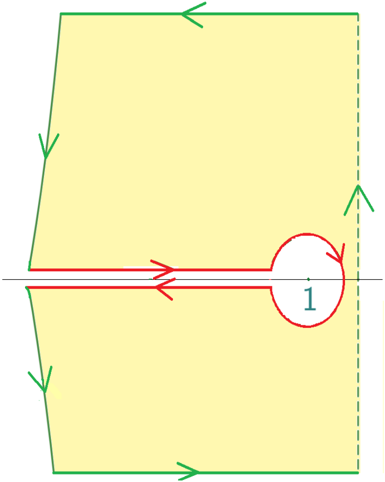

The modification

We identify our sum

$$\sum_{n\le x} \mathbb{1}_{(f(n),q)=1} \cdot \chi_1(\varphi(n))\chi_2(\sigma(n))\chi_3(\sigma_2(n))$$

as the partial sum of the Dirichlet series

$$F_{\widehat{\chi}}(s) = \sum_{n=1}^{\infty} \frac{\mathbb{1}_{(f(n),q)=1}}{n^s} \cdot \chi_1(\varphi(n))\chi_2(\sigma(n))\chi_3(\sigma_2(n)).$$

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But here

$$F_{\widehat{\chi}}(s) = \left(\prod_{\substack{d\mid q \\ d \text{ sqfree}}} \prod_{\substack{\psi \bmod d \\ \psi \text{ primitive}}} L(s,\psi)^{\gamma(\psi)} \right)^{\alpha(q)c_{\widehat{\chi}}} \cdot G(s)$$

for some well behaved $G(s)$. Here
$c_{\widehat{\chi}} = \mathbb{1}_{(u,Q_0)=1} \cdot \chi_1(u-1)\chi_2(u+1)\chi_3(u^2+u+1) \neq 0.$

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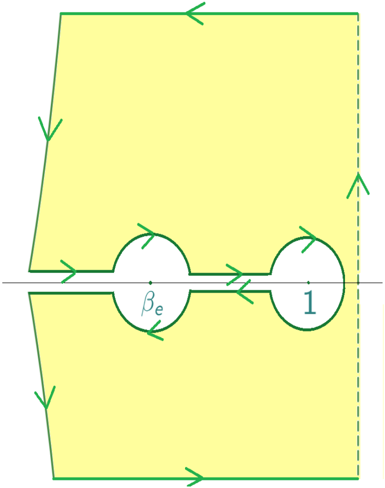

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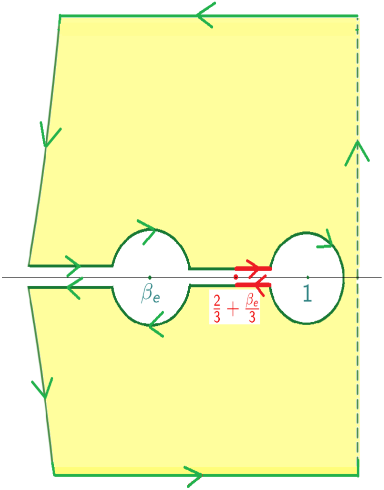

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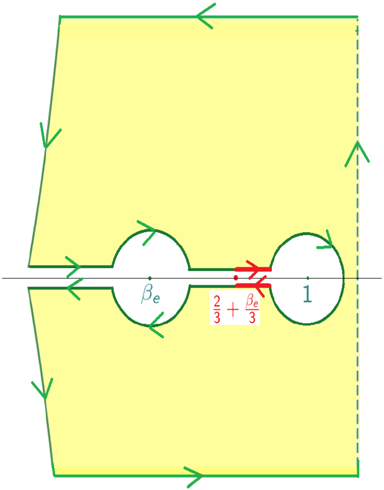
Error terms: Contribution of rest of contour is bounded **very** differently from the usual LSD (inspiration from Scourfield).



After a lot of technical work, we get

$$\sum_{n\le x} \mathbb{1}_{(f(n),q)=1} \cdot \chi_1(\varphi(n))\chi_2(\sigma(n))\chi_3(\sigma_2(n)) \ll \frac{x}{(\log x)^{1-\alpha(q)(c_{\widehat{\chi}}+\delta)}}.$$

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Finally, we obtain the Workhorse Result:

$$\#\{n \le x \ \ \mathrm{conv} : \varphi(n) \equiv a_1, \sigma(n) \equiv a_2, \sigma_2(n) \equiv a_3 \pmod{q}\} \\ \sim \frac{1}{\varphi(q)^3}\#\{n \le x : \gcd(f(n), q) = 1\}. \quad (1)$$

Obtaining Theorems 4 and 5 for $(\varphi, \sigma, \sigma_2)$:

Recall the statements of Theorems 4 and 5:
1. Uniformly in moduli $q \le (\log x)^{(1/2-\epsilon)\alpha(q)}$ s.t. $P^-(q) > 23$,

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$$\#\{n \le x : P_R(n) > q, \varphi(n) \equiv a_1, \sigma(n) \equiv a_2, \sigma_2(n) \equiv a_3 \pmod q\} \sim \frac{1}{\varphi(q)^3} \#\{n \le x : P_R(n) > q, \gcd(\varphi\sigma\sigma_2(n), q) = 1\},$$

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By (1), remains to show that the contribution of inconvenient n is negligible. Need careful arguments studying the anatomy of inconvenient inputs n.

(Some of) the General Main Results

Consider multiplicative functions $f_1, \ldots, f_K : \mathbb{N} \to \mathbb{Z}$ and polynomials $\{W_{i,v}\}_{\substack{1 \leq i \leq K \\ 1 \leq v \leq V}} \subset \mathbb{Z}[T]$, such that $f_i(p^v) = W_{i,v}(p)$.

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Given $k \in \{1, \dots, V\}$, we say that q is **k-admissible** if $\{u \in U_q : (\forall i)\ W_{i,k}(u) \in U_q\} \neq \emptyset$, but $\{u \in U_q : (\forall i)\ W_{i,v}(u) \in U_q\} = \emptyset$, for each $1 \le v \le k-1$.

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Theorem 8 (Narkiewicz, 1982).

Fix a k-admissible integer q. The functions $f_1, \ldots, f_K$ are jointly WUD mod q iff q satisfies **Property N***:*

For every tuple $\widehat{\chi} = (\chi_1, \ldots, \chi_K) \neq (\chi_0, \ldots, \chi_0)$ mod q s.t. $\chi_0(u) \prod_{i=1}^K \chi_i(W_{i,k}(u)) = 1$ on its support, a certain "local factor" associated to $\widehat{\chi}$ vanishes.

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Let $\mathcal{Q}(k; f_1, \cdots, f_K)$ be the set of k-admissible q satisfying Property N.

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Theorem 9 (S.R., 2023).

Fix $\epsilon \in (0,1)$. Under H_1 and H_2, the functions $f_1, \dots, f_K$ are jointly WUD, uniformly modulo $q \in \mathcal{Q}(k; f_1, \cdots, f_K)$, provided any **one** *of the following holds.*

(i) *Either $K = 1$ and $W_{1,k} = W_k$ is linear, or $K \ge 2$, $q \le (\log x)^{(1-\epsilon)\alpha_k(q)/(K-1)}$ and at least one of $\{W_{i,k}\}_{i=1}^{K}$ is linear.*

(ii) *$D_{min} > 1$ and $q \le (\log x)^{(1-\epsilon)\alpha_k(q)(K-1/D_{min})^{-1}}$.*

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Optimality: This result is essentially optimal in the range and arithmetic restrictions on q as well as in the hypotheses H_1 and H_2.

As for $\varphi, \sigma, \sigma_2$, we need to restrict our input sets to get complete uniformity up to arbitrary powers of $\log x$. Fix $K_0 > 1$.

Theorem 10 (S.R., 2023).

Under H_1 and H_2, we have, uniformly in $q \le (\log x)^{K_0}$ lying in $\mathcal{Q}(k; f_1, \cdots, f_K)$ and in $a_1, \dots, a_K \in U_q$,

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1. $R = \max\{k(KD+1), k(1+(k+1)(K-1/D))\}$ *for general q.*
2. *If q is squarefree and $k \ge 2$, then*

$$R = \begin{cases} k(Kk+K-k)+1, & \text{if one of } \{W_{i,k}\}_{i=1}^{K} \text{ not sqfull.} \\ k(Kk+K-k+1)+1, & \text{in general.} \end{cases}$$

3. *If q is squarefree and $k = 1$, then $R = 2K+1$.*
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Optimality: Most of these R's are either exactly or nearly optimal, ensuring joint WUD among as large a set of inputs as possible.

Thank you for your attention!