The Landau-Selberg-Delange method for Dirichlet L-functions, and applications

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Extension? Assume that $\sum_n a_n/n^s = \left(\prod_{\chi \bmod q} L(s,\chi)^{\alpha_\chi}\right) \cdot G(s)$ for all $\mathrm{Re}(s) > 1$, where $\{\alpha_\chi\}_\chi \subset \mathbb{C}$ and $G(s)$ is "well-behaved".

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- Give as **precise** asymptotic series estimating $\sum_{n\leq x} a_n$, **uniformly in q in a wide range** (wider than from previous literature).

One of the main results

Theorem 1 (S.R., outline).
Fix $\nu \in \mathbb{R}^+$. *Assume* $\sum_n a_n/n^s = \left(\prod_{\chi \bmod q} L(s\nu, \chi)^{\alpha_\chi}\right) \cdot G(s)$ *for all* s *with* $\mathrm{Re}(s) > 1/\nu$, *where* $\{\alpha_\chi\}_\chi \subset \mathbb{C}$, *and* $G(s)$ *is some "well-behaved" function. Then for any fixed* $K_0 > 0$, *we have*

$$\sum_{n \le x} a_n = \frac{x^{1/\nu}}{(\log x)^{1-\alpha_{\chi_0}}} \sum_{0 \le j \le N} \frac{\kappa_j}{(\log x)^j} \; + \; O(\text{Error Terms}),$$

uniformly in $x \geq 3$, $N \geq 0$ *and* $q \leq (\log x)^{K_0}$.

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Wider ranges under Landau-Siegel zeros conjecture and GRH.

Some applications

(1) Positive integers with prime divisors restricted to residue classes:
Given $q \in \mathbb{Z}^+$ and $\mathcal{A} \subset (\mathbb{Z}/q\mathbb{Z})^\times$, estimate
$\#\{n \le x : \ p \mid n \implies p \bmod q \in \mathcal{A}\}$.

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- Sathe-Selberg: Local distributions of the functions $\omega(n) = \sum_{p|n} 1$ and $\Omega(n) = \sum_{p|n} v_p(n)$.

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- Theorem 1 $\implies$ Local distributions of $\omega_a(n)$ and $\Omega_a(n)$, uniformly in $q \le (\log x)^{K_0}$ and in $a \in U_q$.

More applications

(3) Residue-class distribution of multiplicative functions: Given multiplicative functions $f_1, \dots, f_K : \mathbb{Z}^+ \to \mathbb{Z}$, estimate $\#\{n \le x : (\forall i)\, f_i(n) \equiv a_i \pmod{q}\}$ for units a_i mod q.

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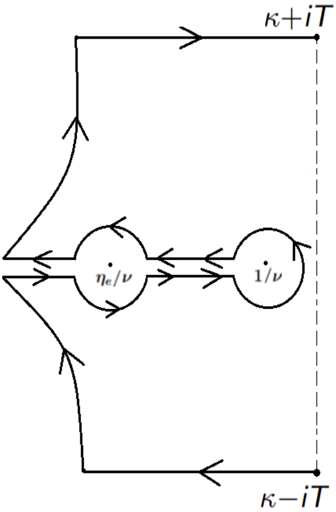
  ✓Large classes of $f_1, \dots, f_K$ and **fixed** q.
- Theorem 1 $\Longrightarrow$ ✓ Extends all known results to $q \le (\log x)^{K_0}$.
  ✓ Qualitatively + Quantitatively optimal results.

Main ideas behind Theorem 1: Summarized

Let $\mathcal{F}(s) = \prod_{\chi \bmod q} L(s\nu, \chi)^{\alpha_\chi}$. Then

$$\sum_{n \le x} a_n \approx \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} \frac{\mathcal{F}(s) G(s) x^s}{s} \, ds$$

(Averaging + pigeonhole argument to control Perron error for a general $\nu > 0$.)



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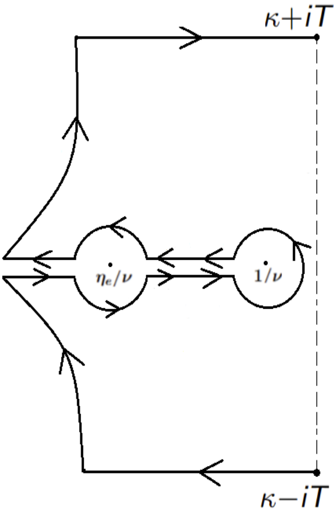
where Γ is the solid contour.



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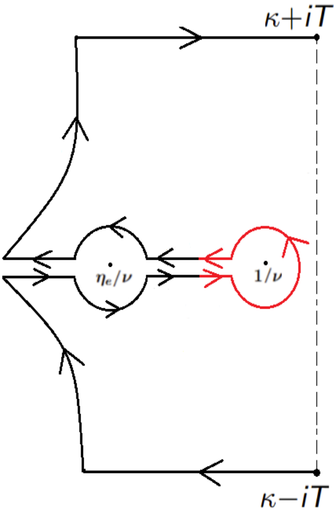
$$\int_{\text{Red part}} \frac{\mathcal{F}(s)G(s)x^s}{s}\, ds$$

$\longrightarrow$ main term, secondary term, ...

Need: Bound on $|\mathcal{F}(s)|$ on the rest of Γ.



Bounding $\mathcal{F}(s) = \prod_{\chi \bmod q} L(s\nu, \chi)^{\alpha_\chi}$ on the rest of Γ

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  (i) Dirichlet polynomial involving $\left\{ \sum_{\chi \bmod q} \alpha_\chi \cdot \chi(n) \right\}_{n=1}^{\infty}$ (weighted)
  $\longrightarrow$ Bounded in terms of λ_q.

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    $\longrightarrow$ Split into regions + (log-free) zero-density estimates

    + various manipulations with Stieltjes integrals.

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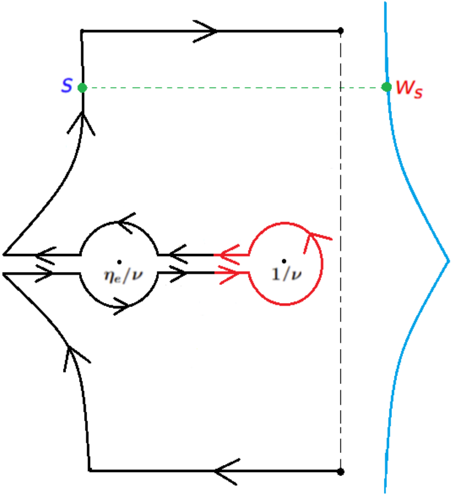
$$\mathcal{H}(s) = \mathcal{F}(s)\,(s - 1/\nu)^{\alpha_{\chi_0}}\,(s - \eta_e/\nu)^{-\alpha_{\chi_e}}$$



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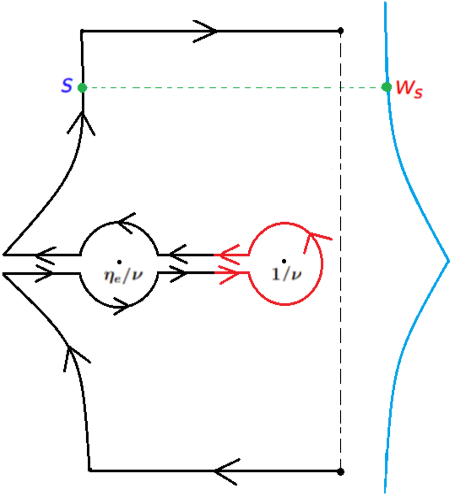
- Relate values of $\log \mathcal{F}(s)$ on Γ with values of $\log \mathcal{F}(w_s)$.
- Dirichlet series for $\log \mathcal{F}(w_s)$.



Thank you for your attention!

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