ANATOMICAL MEAN VALUE BOUNDS ON MULTIPLICATIVE FUNCTIONS AND THE DISTRIBUTION OF THE SUM OF DIVISORS

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Abstract. By suitably combining ideas from the "anatomy of integers" with methods from classical analytic number theory, we establish a new bound on mean values of multiplicative functions under very natural and general hypotheses. This appears to be one of the first results of its kind as it is able to detect crucial power savings missed in known results in the literature, whilst being highly uniform in its parameters and containing only flexible error terms that can be bounded easily in applications. Using this result, we study the distribution of the sum-of-divisors function $\sigma(n)$ in coprime residue classes to moduli $q \le (\log x)^K$, extending results of Śliwa (who had studied this problem for fixed moduli) and obtaining essentially best possible qualitative and quantitative analogues of the Siegel–Walfisz theorem for primes in progressions, with primes replaced by values of $\sigma(n)$. We are able to obtain precise asymptotics for how often $\sigma(n)$ lands in a given coprime residue class mod q, that are uniform in a wide range of q as well as optimal in the error term, the arithmetic restrictions on q and in various other parameters. As a consequence of our results, we establish that the values of $\sigma(n)$ sampled over $n \le x$ with $\sigma(n)$ coprime to q are asymptotically equidistributed among the coprime residue classes mod q, uniformly for odd $q \le (\log x)^K$. On the other hand, if q is even and not divisible by 3, then equidistribution is possible only when we restrict to inputs n having six (or for squarefree q, four) prime factors exceeding q.

1. Introduction

Mean values of multiplicative functions has been a central area of research and a topic of ardent interest in analytic and multiplicative number theory. Two of the classical results in the subject are the theorem of Halász [17] (see [40, Corollary III.4.12] for a precise version) which provides a highly general upper bound on the partial sums of any multiplicative function taking values in the (complex) unit disk, and the analytic method of Landau–Selberg–Delange (see [40, Chapter II.5]) which provides a highly precise asymptotic series for the partial sums of a multiplicative function whose Dirichlet series behaves like a power of the Riemann zeta function. There is a rich variety of literature filling this wide spectrum from generality to precision, with predominant authors such as de la Bretèche, Granville, Hall, Harper, Koukoulopoulos, Montgomery, Soundararajan, Tenenbaum, Vaughan and others [18, 19, 20, 36, 40, 24, 16, 41, 13, 14, 15, 8] studying the problem of estimating mean values of multiplicative functions taking values in the unit disk. However, despite the extensive amount of literature on the subject, situations still arise where known estimates either turn out to not be precise enough, or come with additional error terms that become too large in applications to yield any useful information, or contain unmanageable expressions that simply cannot be bounded in practice.

2020 *Mathematics Subject Classification.* Primary 11A25; Secondary 11N36, 11N37, 11N64, 11N69.

Key words and phrases. multiplicative function, mean values, equidistribution, uniform distribution, weak equidistribution, weak uniform distribution, sum of divisors.

One such situation is in the study of the distribution of values of arithmetic functions in residue classes to uniformly varying moduli. While this question has been thoroughly studied for fixed moduli, the first steps towards obtaining results where the moduli are allowed to vary in a wide range seem to have been taken in [22, 31, 32]. However despite the panoply of mean value estimates in the literature, the lack of a suitable estimate that is both sufficiently precise and uniform compelled the authors to largely avoid multiplicative machinery and resort to more 'quasielementary' combinatorial methods. One downside of this was that it seemed somewhat hopeless to get a satisfactory estimate for how often a multiplicative function takes values from a given residue class when the modulus is allowed to vary in a wide range. Furthermore, these results and methods did not have satisfactory applications to various commonly-studied arithmetic functions.

In recent joint work with Pollack [30], we took the first steps towards (partially) addressing the first of these issues: We gave a new upper bound on the partial sum of a multiplicative function taking values in the unit disk. Using this result, we were able to quantitatively study the distribution of the Alladi-Erdős function $A(n) := \sum_{p^k \| n} k \cdot p$ and Euler's totient function $\varphi(n) := \#\{1 \leq d \leq n : \gcd(d, n) = 1\}$ in residue classes to moduli that were allowed to vary uniformly in a wide range. As a consequence, we were able to deduce the equidistribution of these functions in residue classes to varying moduli (interpreted appropriately). Our results extended those of Goldfeld [12] and Narkiewicz [25] who had studied the corresponding questions for fixed moduli.

Some closely related and prevalent multiplicative functions are the sum-of-divisors function $\sigma(n) = \sum_{d|n} d$ and the "sum-of-divisor powers" functions $\sigma_\nu(n) = \sum_{d|n} d^\nu$. The distribution of these functions to fixed moduli has received immense interest in the literature. Śliwa [39] obtains a necessary and sufficient criterion for $\sigma(n)$ to be equidistributed among the coprime residue classes mod q, and his result was extended (either completely or partially) to the functions $\sigma_\nu(n)$ by Narkiewicz, Rayner, Dobrowolski, Fomenko and others; see [10], [29], [26], [27, Theorem 6.12], [34], [35]. However, none of the results mentioned so far or the methods used in them (including work in [30]) are able to give satisfactory qualitative or quantitative results on the distribution of $\sigma(n)$ in coprime residue classes to varying moduli.

Motivated by these problems, our first main result in this manuscript is a variant of the main result Theorem 1.1 in [30]. We give a sharp yet flexible upper bound on the mean values of a multiplicative function, that can detect certain crucial power savings missed in previous literature, whilst remaining highly uniform in its parameters and containing only error terms that are easy to bound and optimize. In what follows, $\mathbb{U} = \{s \in \mathbb{C} : |s| \leq 1\}$ denotes the unit disk in the complex plane, $\pi(y)$ is the number of primes up to y, and $\Psi(x, z)$ denotes the number of z-smooth positive integers up to x (i.e., those that have no prime factor exceeding z). The quantity $\Psi(x, z)$ has been studied extensively in the literature (see for instance, [40, Chapter III.5]).

Theorem 1.1. *Let $f \colon \mathbb{N} \to \mathbb{U}$ be a multiplicative function and x, y, z, M be positive real numbers such that $M \geq 1$, $1 < z < x$ and $e^{11/2} \leq y \leq z^{1/(18 \log\log z)^2}$. Assume that there exists*

$\varrho \in \mathbb{U}$ and a decreasing function $\mathcal{E}\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\lim_{X\to\infty} \mathcal{E}(X) = 0$, such that

$$\sum_{y<p\le Y} f(p) = \varrho(\pi(Y) - \pi(y)) + O(MY\mathcal{E}(y)) \tag{1}$$

for all $Y \ge y$. Then we have the following upper bound

$$\sum_{n\le x} f(n) \ll \frac{|\varrho| x}{\log z}\left(\frac{\log z}{\log y}\right)^{\mathrm{Re}(\varrho)} \exp\left(\sum_{p\le y}\frac{|f(p)|}{p}\right) + \frac{x(\log y)^{1+|\varrho|}}{(\log z)^{2-\mathrm{Re}(\varrho)}}\exp\left(\sum_{p\le y}\frac{|f(p)|}{p}\right) + \Psi(x,z) + Mx(\log x)^2\left(\mathcal{E}(y) + \frac{1}{y}\right), \tag{2}$$

where the implied constant depends at most on the implied constant in (1).

A few comments are in order. The assumption (1) says that f behaves like a constant on average at the primes, which is a very standard hypothesis in several mean value estimates. The parameters y and z can be chosen quite freely in applications since the technical conditions on them are usually quite easy to satisfy. This enhances the flexibility of our bounds.

The crucial new ingredient in this bound is the blending of combinatorial ideas belonging to the realm of the "anatomy of integers" with classical analytic methods from multiplicative number theory. While most mean value estimates in the literature rely predominantly on the latter, the proof of the above theorem also relies heavily on the former: We split off the parts of our inputs n that are divisible by very large primes, so as to essentially rewrite the sum $\sum_{n\le x} f(n)$ in terms of a multiplicative function exhibiting high cancellation. However at the same time, in order to detect this sufficient cancellation, we do crucially require "pure analytic" inputs coming from an explicit version of the Landau–Selberg–Delange method enunciated by Chang and Martin [6].

In applications, it is usually the sizes of the first two terms that determines the usefulness of the bound (2); this is because for many natural choices of y and z, the terms involving $\Psi(x,z)$, $\mathcal{E}(y)$ and $1/y$ in (2) typically become very small. The crucial difference between this bound and the one in Theorem 1.1 of [30] is the factor $(\log z)^{\mathrm{Re}(\varrho)}$ in the first expression, whose analogue in [30, Theorem 1.1] contained the larger factor $(\log x)^{|\varrho|}$. These greater savings turn out to play a crucial role in certain applications. However, our bound above cannot entirely subsume [30, Theorem 1.1] because the additional term with $(\log y)^{1+|\varrho|}$ (which has no analogue in [30, Theorem 1.1]) can sometimes limit the applicability of our bound. We shall witness these phenomena in our applications (see the remark following the proof of Proposition 5.4).

Both Theorem 1.1 and its arguments can be used to study the asymptotic distribution of multiplicative functions in coprime residue classes to moduli varying uniformly in a wide range. Here it is important to clarify that the naive notion of "uniform distribution in residue classes" [1] is **not** the correct notion to work with: For instance, it follows from classical results that the Euler totient function $\varphi(n)$ is almost always divisible by any fixed integer q and the same is true for $\sigma(n)$. (This means that $\varphi(n)$ and $\sigma(n)$ are divisible by q for all $n \le x$ with at

[1] We say that an integer-valued arithmetic function g is uniformly distributed (or equidistributed) modulo q if $\#\{n \le x : g(n) \equiv b \pmod q\} \sim x/q$ as $x \to \infty$, for each residue class b mod q.

most $o(x)$ exceptions, as $x \to \infty$.) Hence neither $\varphi(n)$ nor $\sigma(n)$ is equidistributed modulo any given $q > 1$. Motivated by this, Narkiewicz in [25] introduces the notion of weak uniform distribution: Given an integer-valued arithmetic function f and a positive integer q, we say that f is weakly uniformly distributed (or weakly equidistributed or WUD) modulo q if there are infinitely many positive integers n for which $\gcd(f(n), q) = 1$, and if

$$\#\{n \le x : f(n) \equiv a \pmod q\} \sim \frac{1}{\varphi(q)} \#\{n \le x : \gcd(f(n), q) = 1\}, \quad \text{as } x \to \infty, \tag{3}$$

for each coprime residue class a mod q.

In [25, Theorem I], Narkiewicz gives a general criterion for deciding weak equidistribution in a large class of a multiplicative functions. An application of this criterion allows us to deduce that $\sigma(n)$ is weakly equidistributed mod q for any q coprime to 6. This was improved to a complete characterization by Śliwa [39] (see also [27, Proposition 7.9, p. 106]), who showed that $\sigma(n)$ is weakly equidistributed *precisely* modulo those fixed q that are not multiples of 6. It still remains a highly nontrivial problem to explicitly characterize for a given f, the set of q for which f is weakly equidistributed mod q; Narkiewicz's monograph [27, Chapter VI] contains an algorithmic solution to this problem in certain special cases.

In all these results, the modulus q was assumed to be fixed. A natural and interesting question is whether weak equidistribution continues to hold as q varies uniformly in a suitable range depending on the stopping point x of inputs. In other words, we seek analogues of the Siegel–Walfisz theorem for primes in arithmetic progressions, but with primes replaced by values of multiplicative functions. To this end, we say that an integer-valued arithmetic function $f(n)$ is weakly equidistributed (or WUD) mod q, uniformly for $q \le (\log x)^K$, if:

(i) For every such q, there are infinitely many n for which $f(n)$ is coprime to q, and

(ii) The relation (3) holds uniformly in moduli $q \le (\log x)^K$ and in coprime residue classes a mod q. Explicitly, this means that for any $\epsilon > 0$, there exists $X(\epsilon) > 0$ such that the ratio of the left hand side of (3) to the right lies in $(1-\epsilon, 1+\epsilon)$ for all $x > X(\epsilon)$, $q \le (\log x)^K$ and coprime residues a mod q.

The weak equidistribution of certain classes of arithmetic functions to uniformly varying moduli appears to have been first investigated in [22, 31, 32] via combinatorial methods. A general theorem in [32] shows that $\sigma(n)$ is weakly equidistributed uniformly to moduli $q \le (\log x)^K$ coprime to 6. However, the methods in [32] are unable to address the case when our varying q is odd but divisible by 3 as well as the case when q is even but not divisible by 3 (while Śliwa is able to show that $\sigma(n)$ is weakly equdistributed modulo any such fixed q). In addition, neither these combinatorial methods nor any mean value results from other papers mentioned so far are able to say anything useful regarding how often $\sigma(n)$ lands in a given coprime residue class mod q in our desired range of q, since they either give extremely weak error terms, or miss crucial power savings in certain naturally-arising character sums, or give rise to additional error terms that are so rapidly growing in q that they severely impede uniformity in q. As such, all this prior work is still significantly far away from a suitable analogue of the Siegel–Walfisz theorem for $\sigma(n)$.

Our next three theorems address these defects and also reveal some surprising phenomena for even moduli. As applications of Theorem 1.1 and the methods used to prove it, we are (for the first time) able to detect the crucial power savings missed before and obtain essentially best possible qualitative and quantitative analogues of the Siegel–Walfisz theorem, thus extending Śliwa's results as well: We give sharp asymptotics (with error terms that can be expected to be essentially of the correct orders of magnitude) for how often $\sigma(n)$ takes values from a given coprime residue class mod q, where q also varies in a wide range and satisfies optimal arithmetic restrictions. Related counting problems for $\sigma(n)$, $\varphi(n)$, the sum-of-aliquot-divisors" function $s(n) := \sigma(n) - n$ and the "cototient function" $\beta(n) = n - \varphi(n)$ have also been investigated in [9, 5, 2, 1, 22, 33, 11, 28, 21].

We first state the promised result for odd q. In what follows, we set

$$\alpha(q) := \frac{1}{\varphi(q)} \#\{v \bmod q : \gcd(v(v+1), q) = 1\} = \prod_{\ell \mid q} \left(1 - \frac{1}{\ell - 1}\right),$$

the last equality being true by the Chinese Remainder Theorem. Here, the expression $v+1$ arises from the polynomial $T+1$ that controls the behavior of $\sigma(n)$ at the primes.

Theorem 1.2. *Fix $K > 0$ and $\epsilon \in (0,1)$. We have*

$$\#\{n \le x : \sigma(n) \equiv a \pmod q\} = \frac{1}{\varphi(q)} \#\{n \le x : \gcd(\sigma(n), q) = 1\} + O_{K,\epsilon}\left(\frac{x}{\varphi(q)(\log x)^{1-\alpha(q)(1/3+\epsilon)}}\right),$$

uniformly in odd moduli $q \le (\log x)^K$ and coprime residue classes a mod q. As a consequence, $\sigma(n)$ is weakly equidistributed mod q, uniformly for odd moduli $q \le (\log x)^K$.

To extend Śliwa's results and obtain a complete analogue of the Siegel–Walfisz theorem for $\sigma(n)$, we still need to address the case when the modulus q is even. Here there are two new difficulties that arise. First, the set of relevant inputs $n \le x$ is highly sparse. In fact, elementary number theory shows that if $\sigma(n)$ is odd, then n must be of the form $2^k m^2$ for some odd m, so that there are $O(x^{1/2})$ many $n \le x$ which have $\sigma(n)$ coprime to a given even modulus q. Sparse sets like this can often present difficulties while studying arithmetic questions about them. However, we can work around this sparsity issue by looking at the behaviour of $\sigma(n)$ at the squares of primes.

Another crucial difficulty is that for even q, inputs n with too few large prime factors present obstructions to uniformity in $q \le (\log x)^K$. In order to restore this uniformity, it becomes necessary to restrict the set of n to those having sufficiently many prime factors exceeding q. To make this precise, we write $P(n)$ or $P^+(n)$ for the largest prime divisor of n, setting $P(1) = 1$. With $P_1(n) := P(n)$, we inductively define $P_k(n) := P_{k-1}(n/P(n))$. Thus, $P_k(n)$ is the kth largest prime factor of n (counted with multiplicity), with $P_k(n) = 1$ if $\Omega(n) < k$.

In what follows, we set

$$\widetilde{\alpha}(q) := \frac{1}{\varphi(q)} \#\{v \bmod q : \gcd(v(v^2+v+1), q) = 1\} = \prod_{\substack{\ell \mid q \\ \ell \equiv 1 \ (\mathrm{mod}\ 3)}} \left(1 - \frac{2}{\ell - 1}\right),$$

the last equality being true by the Chinese Remainder Theorem and the law of quadratic reciprocity. Here, the expression v^2+v+1 arises from the polynomial T^2+T+1 that controls the behavior of $\sigma(n)$ at the squares of primes.

Theorem 1.3. *Fix $K > 0$ and $\epsilon \in (0,1)$. We have*

$$\begin{aligned}(4)\quad &\#\{n \le x : P_6(n) > q,\ \sigma(n) \equiv a \pmod q\}\\ &\qquad = \frac{1}{\varphi(q)}\#\{n \le x : P_6(n) > q,\ \gcd(\sigma(n), q) = 1\} + O_{K,\epsilon}\left(\frac{x^{1/2}}{\varphi(q)(\log x)^{1-\widetilde{\alpha}(q)(1/4+\epsilon)}}\right),\end{aligned}$$

uniformly in even moduli $q \le (\log x)^K$ not divisible by 3, and in coprime residues a mod q. Hence as $x \to \infty$, we have in the same range of uniformity in q,

(5)
$$\#\{n \le x : P_6(n) > q,\ \sigma(n) \equiv a \pmod q\} \sim \frac{1}{\varphi(q)}\#\{n \le x : P_6(n) > q,\ \gcd(\sigma(n), q) = 1\}.$$

In subsection 6.1, we give an explicit counterexample showing that the restriction $P_6(n) > q$ is optimal in the sense that it cannot be replaced by a condition of the form "$P_k(n) > q$" with $k < 6$, while still retaining weak equidistribution among the corresponding set of n's in the same range of uniformity in q. In fact, we will show that uniformity fails to moduli $q \le (\log x)^K$ that are of the form $2Q^2$ for certain odd squarefree integers Q having several prime factors. It is quite intriguing to note that the proof of Theorem 1.3 and the counterexample showing the optimality of the condition $P_6(n) > q$ seem to have somewhat different roots: While the proof relies heavily on certain character sum bounds modulo prime powers, the aforementioned counterexample relies on an excess in the number of lifts of solutions to a bivariate polynomial congruence from prime moduli to prime square moduli. These excess lifts in turn come from the $\mathbb{F}_\ell$-rational points of certain affine irreducible curves over the finite field $\mathbb{F}_\ell$.

Now if we restrict our attention to *squarefree* moduli q, then it turns out that we can enlarge the set of inputs n to those having four (as opposed to six) prime divisors exceeding q.

Theorem 1.4. *Fix $K > 0$ and $\epsilon \in (0,1)$. We have*

$$\begin{aligned}(6)\quad &\#\{n \le x : P_4(n) > q,\ \sigma(n) \equiv a \pmod q\}\\ &\qquad = \frac{1}{\varphi(q)}\#\{n \le x : P_4(n) > q,\ \gcd(\sigma(n), q) = 1\} + O_{K,\epsilon}\left(\frac{x^{1/2}}{\varphi(q)(\log x)^{1-\widetilde{\alpha}(q)(1/4+\epsilon)}}\right),\end{aligned}$$

uniformly in squarefree even moduli $q \le (\log x)^K$ not divisible by 3, and in coprime residue classes a mod q. Hence as $x \to \infty$, we have in the same range of uniformity in q,

(7)
$$\#\{n \le x : P_4(n) > q,\ \sigma(n) \equiv a \pmod q\} \sim \frac{1}{\varphi(q)}\#\{n \le x : P_4(n) > q,\ \gcd(\sigma(n), q) = 1\}.$$

In subsection 7.1, we show that the restriction $P_4(n) > q$ is optimal for squarefree even q in the same sense as in the previous result. Moreover, the error terms in Theorems 1.2 through 1.4 can be expected to be essentially best possible, as suggested by the following heuristic: In order to detect the congruence $\sigma(n) \equiv a \pmod q$, the primary objects of study are the multiplicative functions $\chi \circ \sigma$ for nonprincipal Dirichlet characters χ mod q. With these functions playing the role of "f" in Theorem 1.1, the role of the parameter "ϱ" is played by the averages

$$\rho_\chi := \frac{1}{\varphi(q)} \sum_{\substack{v \bmod q \\ \gcd(v,q)=1}} \chi(v+1) \quad \left(\text{resp. } \eta_\chi := \frac{1}{\varphi(q)} \sum_{\substack{v \bmod q \\ \gcd(v,q)=1}} \chi(v^2+v+1)\right)$$

in Theorem 1.2 (resp., Theorems 1.3 and 1.4). It turns out that the maximum values of the real parts of ρ_χ (resp. η_χ) are precisely the constants "1/3" (resp. "1/4") appearing in the exponent of $\log x$ (in the error terms). In fact, these values are attained by characters χ of conductor 15 (resp. conductors 5, 7, 13, 35), in the cases when q is divisible by the respective conductors (see the remarks at the end of the proofs of Theorem 1.2 and Proposition 5.1).

We are not sure what to conjecture regarding the best possible range of uniformity in q in Theorems 1.2 to 1.4. By modifying the arguments towards the end of the introduction in [22], standard conjectures on shifted primes without large prime factors imply that, even if we were to restrict to *prime* moduli q, we cannot replace $(\log x)^K$ by $L(x)^{1+\delta}$ for any $\delta > 0$, where $L(x) = x^{\log\log\log x/\log\log x}$.

We conclude this introductory section with two remarks. First, in order to deduce the weak equidistribution of $\sigma(n)$ from the respective quantitative statements in Theorems 1.2-1.4, highly crude estimates on the main terms such as those obtained from Proposition 4.1 below turn out to be sufficient. Nonetheless, more precise asymptotic formulas for the main terms in these theorems can be readily deduced from Theorems A and B in work of Scourfield [38].

Finally, as an immediate byproduct of our arguments, we can obtain sharp upper bounds on the character sums $\sum_{n\le x} \chi(\sigma(n))$, which (in cases of significance) appear to have the correct orders of magnitude. Such sums are interesting in their own right as they have been ardently studied by several authors: For instance, Balasuriya, Shparlinski and Sutantyo in [3] derive upper bounds on very similar character sums involving the Euler totient function $\varphi(n)$, and while it is easy to see that their work also goes through for $\sigma(n)$ replacing $\varphi(n)$, the bounds we obtain are much sharper in the range $q \le (\log x)^K$. Banks and Shparlinski ([4] and [5]) also study *exponential* sums involving $\sigma(n)$ (see the remarks at the end of both these papers), while Balasuriya, Luca, Banks and Shparlinski (see [2] and [1]) investigate exponential and character sums for the sum-of-aliquot-divisors function $s(n) = \sum_{d\mid n:\ d\ne n} d$.

Notation and conventions: To us, the zero function is not multiplicative (thus, $f(1) = 1$ for any multiplicative function f). We denote the largest prime divisor of n by $P^+(n)$ or $P(n)$, the k-th largest prime divisor of n (counted with multiplicity) by $P_k(n)$, and the least prime divisor of n by $P^-(n)$. Throughout, the letters p and ℓ shall denote primes, and U_q the group of units (or the multiplicative group) modulo q. When there is no danger of confusion, we shall abbreviate "$\gcd(a,b)$" to "(a,b)". Implied constants in $\ll$ and O-notation are allowed to depend on any parameters declared as "fixed"; other dependence will be noted explicitly (for example, with subscripts). We write $\log_k$ for the kth iterate of the natural logarithm.

2. Uniform bounds on the partial sums of multiplicative functions: Proof of Theorem 1.1

Our arguments for Theorems 1.1 and 1.2 begin similarly to those given for [30, Theorems 1.1 and 1.3], but we include the complete argument here in order to keep the exposition self-contained. We first bound the contribution of the $n \le x$ that are either z-smooth or have a repeated prime factor exceeding y. Since $|f(n)| \le 1$, the contribution of the former n has absolute value at most $\Psi(x, z)$, while that of the latter n is bounded in absolute value by

$$\sum_{p>y} \sum_{\substack{n \le x \\ p^2 | n}} 1 \le x \sum_{p>y} \frac{1}{p^2} \ll \frac{x}{y}.$$

Both of these are absorbed in the expressions given in the claimed bounds.

Let $\sum_{n \le x}^* f(n)$ denote the sum over the remaining n, namely those that have $P^+(n) > z$ and no repeated prime factor exceeding y. Any n counted in this sum can be uniquely written in the form $mP_j \cdots P_1$ for some $j \ge 1$, where $P_1 = P^+(n) > z$ and $P^+(m) \le y < P_j < \cdots < P_1$. As such $f(n) = f(m)f(P_j) \cdots f(P_1)$ and

$$\begin{aligned} \sum\nolimits_{n \le x}^* f(n) &= \sum_{j \ge 1} \sum_{\substack{m \le x \\ P^+(m) \le y}} f(m) \sum_{\substack{P_1, \ldots, P_j \\ P_1 > z,\ P_j \cdots P_1 \le x/m \\ y < P_j < \cdots < P_1}} f(P_j) \cdots f(P_1) \\ &= \sum_{j \ge 1} \sum_{\substack{m \le x \\ P^+(m) \le y}} f(m) \sum_{\substack{P_2, \ldots, P_j \\ P_j \cdots P_2 \le x/mz \\ y < P_j < \cdots < P_2}} f(P_j) \cdots f(P_2) \sum_{\max\{P_2, z\} < P_1 \le x/mP_2 \cdots P_j} f(P_1). \end{aligned}$$

Here $\max\{P_2, z\}$ is to be replaced by z in the case $j = 1$.

Employing (1) to estimate the innermost sum on P_1, we find that

$$\sum\nolimits_{n \le x}^* f(n) = \varrho \sum_{j \ge 1} \frac{1}{(j-1)!} \sum_{\substack{m \le x \\ P^+(m) \le y}} f(m) \sum_{z < P_1 \le x/my^{j-1}} \sum_{\substack{P_2, \ldots, P_j \in (y, P_1) \\ P_2, \ldots, P_j \text{ distinct} \\ P_2 \cdots P_j \le x/mP_1}} f(P_2) \cdots f(P_j) + O(Mx\mathcal{E}(y) \log x), \tag{8}$$

where we have observed that the total size of the resulting error term incurred upon an application of (1) is

$$\ll Mx\mathcal{E}(y) \sum_{j \ge 1} \sum_{\substack{m, P_2, \ldots, P_j \\ mP_2 \cdots P_j \le x/z \\ P^+(m) \le y < P_j < \cdots < P_2}} \frac{1}{mP_2 \cdots P_j} \le Mx\mathcal{E}(y) \sum_{n \le x/z} \frac{1}{n} \ll Mx\mathcal{E}(y) \log x.$$

Now for $j \geq 2$ and each $i \in \{2, \ldots, j\}$, estimate (1) shows that

(9)
$$\sum_{\substack{y<P_i\le x/mP_1\cdots P_{i-1}P_{i+1}\cdots P_j \\ P_i<P_1,\ r\neq i \implies P_r\neq P_i}} f(P_i) = \sum_{\substack{y<P_i\le x/mP_1\cdots P_{i-1}P_{i+1}\cdots P_j \\ P_i<P_1}} f(P_i) + O(j)$$
$$= \varrho \sum_{\substack{y<P_i\le x/mP_1\cdots P_{i-1}P_{i+1}\cdots P_j \\ P_i<P_1,\ r\neq i \implies P_r\neq P_i}} 1 + O\left(j + \frac{Mx}{mP_1\cdots P_{i-1}P_{i+1}\cdots P_j}\mathcal{E}(y)\right).$$

For each $j \geq 2$, we use this estimate for $i \in \{2, \ldots, j\}$ in order to successively remove the $f(P_2), \ldots, f(P_j)$ occurring in the main term of (8). To this end, we define

$$\widetilde{\mathcal{E}} := j \sum_{\substack{P_2,\ldots,P_{j-1}\in(y,P_1) \\ P_2,\ldots,P_{j-1}\text{ distinct} \\ P_2\cdots P_{j-1}\le x/myP_1}} 1 + \frac{Mx\mathcal{E}(y)}{mP_1} \sum_{\substack{P_2,\ldots,P_{j-1}\in(y,P_1) \\ P_2,\ldots,P_{j-1}\text{ distinct} \\ P_2\cdots P_{j-1}\le x/myP_1}} \frac{1}{P_2\cdots P_{j-1}},$$

and write

$$\sum_{\substack{P_2,\ldots,P_j\in(y,P_1) \\ P_2,\ldots,P_j\text{ distinct} \\ P_2\cdots P_j\le x/mP_1}} f(P_2)\cdots f(P_j) = \sum_{\substack{P_3,\ldots,P_j\in(y,P_1) \\ P_3,\ldots,P_j\text{ distinct} \\ P_3\cdots P_j\le x/myP_1}} f(P_3)\cdots f(P_j) \sum_{\substack{y<P_2\le x/mP_1P_3\cdots P_j \\ P_2<P_1,\ r\neq 2 \implies P_2\neq P_r}} f(P_2)$$
$$= \varrho \sum_{\substack{P_2,\ldots,P_j\in(y,P_1) \\ P_2,\ldots,P_j\text{ distinct} \\ P_2\cdots P_j\le x/mP_1}} f(P_3)\cdots f(P_j) + O(\widetilde{\mathcal{E}}),$$

where in the last step we have noted that the error term resulting from the application of (9) for $i = 2$ is, by relabelling, equal to $\widetilde{\mathcal{E}}$. Likewise, invoking (9) for $i = 3, \ldots, j$, we obtain

$$\sum_{\substack{P_2,\ldots,P_j\in(y,P_1) \\ P_2,\ldots,P_j\text{ distinct} \\ P_2\cdots P_j\le x/mP_1}} f(P_2)\cdots f(P_j) = \varrho^{j-1} \sum_{\substack{P_2,\ldots,P_j\in(y,P_1) \\ P_2,\ldots,P_j\text{ distinct} \\ P_2\cdots P_j\le x/mP_1}} 1 + O(j\widetilde{\mathcal{E}}).$$

Inserting this into (8) for each $j \geq 2$ incurs a total error of size

$$\ll \sum_{j\ge 2} \frac{1}{(j-1)!} \sum_{\substack{m\le x \\ P^+(m)\le y}} \sum_{z<P_1\le \frac{x}{my^{j-1}}} \left\{ j^2 \sum_{\substack{P_2,\ldots,P_{j-1}\in(y,P_1) \\ P_2,\ldots,P_{j-1}\text{ distinct} \\ P_2\cdots P_{j-1}\le x/myP_1}} 1 + \frac{Mxj\mathcal{E}(y)}{mP_1} \sum_{\substack{P_2,\ldots,P_{j-1}\in(y,P_1) \\ P_2,\ldots,P_{j-1}\text{ distinct} \\ P_2\cdots P_{j-1}\le x/myP_1}} \frac{1}{P_2\cdots P_{j-1}} \right\}$$
$$\ll (\log x)^2 \sum_{j\ge 2} \sum_{\substack{m,P_1,\ldots,P_{j-1} \\ mP_1\cdots P_{j-1}\le x/y \\ P_1>z,\ P^+(m)\le y<P_{j-1}<\cdots<P_1}} 1 \ + \ Mx(\log x)\mathcal{E}(y) \sum_{\substack{m,P_1,\ldots,P_{j-1} \\ mP_1\cdots P_{j-1}\le x/y \\ P_1>z,\ P^+(m)\le y<P_{j-1}<\cdots<P_1}} \frac{1}{mP_1\cdots P_{j-1}}$$
$$\ll (\log x)^2 \sum_{n\le x/y} 1 \ + \ Mx(\log x)\mathcal{E}(y) \sum_{n\le x/y} \frac{1}{n} \ll Mx(\log x)^2 \left(\mathcal{E}(y) + \frac{1}{y}\right).$$

As a consequence, we obtain

(10)

$$\sideset{}{^*}\sum_{n\le x} f(n) = \varrho \sum_{j\ge 1} \frac{\varrho^{j-1}}{(j-1)!} \sum_{\substack{m\le x\\ P^+(m)\le y}} f(m) \sum_{\substack{P_1,\dots,P_j\\ P_1>z,\ P_1\cdots P_j\le x/m\\ P_2\cdots P_j\in(y,P_1)\text{ distinct}}} 1 + O\left(Mx(\log x)^2\left(\mathcal{E}(y)+\frac{1}{y}\right)\right)$$

$$= \sum_{\substack{m\le x/z\\ P^+(m)\le y}} f(m) \sum_{j\ge 1} \sum_{\substack{P_1,\dots,P_j\\ P_1>z,\ P_1\cdots P_j\le x/m\\ y<P_j<\cdots<P_1}} \varrho^j \;+\; O\left(Mx(\log x)^2\left(\mathcal{E}(y)+\frac{1}{y}\right)\right).$$

At this point, we observe that every squarefree positive integer $r \le x/m$ having $P^+(r) > z$ and $P^-(r) > y$ can be uniquely written in the form $P_j \cdots P_1$ with $P_1 > z$ and $y < P_j < \cdots < P_1$, and in this case $j = \Omega(r)$. As such, the main term in the final expression in (10) is equal to

$$\sum_{\substack{m\le x/z\\ P^+(m)\le y}} f(m) \sum_{\substack{r\le x/m\\ r\text{ squarefree}\\ P^+(r)>z,\ P^-(r)>y}} \varrho^{\Omega(r)}\,. \tag{11}$$

Ignoring the condition $P^+(r) > z$ incurs a total error of size

$$\ll \sum_{\substack{m\le x/z\\ P^+(m)\le y}} \sum_{\substack{r\le x/m\\ P^-(r)>y,\, P^+(r)\le z}} 1 \le \sum_{\substack{n\le x\\ P^+(n)\le z}} 1 \le \Psi(x,z),$$

which is absorbed in the bound claimed in the theorem statement. Moreover, since any non-squarefree r having $P^-(r) > y$ is divisible by the square of a prime exceeding y, ignoring the squarefreeness condition in (11) incurs a total error

$$\ll \sum_{\substack{m\le x/z\\ P^+(m)\le y}} \sum_{p>y} \sum_{\substack{r\le x/m\\ p^2\mid r}} 1 \ll x \sum_{\substack{m\le x/z\\ P^+(m)\le y}} \frac{1}{m} \sum_{p>y} \frac{1}{p^2} \ll \frac{x}{y\log y} \prod_{\ell\le y}\left(1+\sum_{v\ge 1}\frac{1}{\ell^v}\right) \ll \frac{x}{y},$$

which is also absorbed in the claimed bound. Hence, up to a negligible error, the expression in (11) is equal to

$$\sum_{\substack{m\le x/z\\ P^+(m)\le y}} f(m) \sum_{r\le x/m} \mathbb{1}_{P^-(r)>y}\ \varrho^{\Omega(r)}. \tag{12}$$

In order to estimate the innermost sum, we shall be making use of the lemma below, which we will establish in the next section.

Lemma 2.1. *Let $X, Y, Z \ge e^{11/2}$ be positive real numbers satisfying $Y \le Z^{1/(18\log\log Z)^2}$ and $Z \le X$. We have for all $\beta \in \mathbb{U}$,*

$$\sum_{n\le X} \mathbb{1}_{P^-(n)>Y}\ \beta^{\Omega(n)} = \frac{X}{(\log X)^{1-\beta}}\left\{\frac{e^{-\gamma\beta}}{\Gamma(\beta)(\log Y)^{\beta}}\left(1+O(\exp(-C_0\sqrt{\log y}))\right) + O\left(\frac{(\log Y)^{1+|\beta|}}{\log Z}\right)\right\},$$

where $C_0 > 0$ is an absolute constant. In particular,

$$\sum_{n\le X} \mathbb{1}_{P^-(n)>Y}\ \beta^{\Omega(n)} \ll \frac{X}{(\log X)^{1-\mathrm{Re}(\beta)}}\left\{\frac{|\beta|}{(\log Y)^{\mathrm{Re}(\beta)}} + \frac{(\log Y)^{1+|\beta|}}{\log Z}\right\}.$$

The implied constants in the above estimates are absolute.

Applying the above lemma with $X := x/m$, $Y := y$, $Z := z$ and $\beta := \varrho$, we find that the expression in (12) is

$$\ll \sum_{\substack{m\le x/z\\ P^+(m)\le y}} |f(m)|\cdot\frac{x/m}{(\log(x/m))^{1-\mathrm{Re}(\varrho)}}\left\{\frac{|\varrho|}{(\log y)^{\mathrm{Re}(\varrho)}} + \frac{(\log y)^{1+|\varrho|}}{\log z}\right\}$$

$$\ll \frac{x}{(\log z)^{1-\mathrm{Re}(\varrho)}}\left(\sum_{m:\,P^+(m)\le y}\frac{|f(m)|}{m}\right)\left\{\frac{|\varrho|}{(\log y)^{\mathrm{Re}(\varrho)}} + \frac{(\log y)^{1+|\varrho|}}{\log z}\right\}.$$

Finally, since $|f(n)|\le 1$ for all n, the sum on m is $\ll \exp(\sum_{p\le y}|f(p)|/p)$, yielding the estimate in the theorem. This completes the proof of Theorem 1.1, up to the proof of Lemma 2.1.

3. Proof of Lemma 2.1: the Landau–Selberg–Delange method

A comprehensive account of the method of Landau–Selberg–Delange may be found in Tenenbaum [40, Chapter II.5]. However, we shall be using a recent formulation of this method due to Chang and Martin [6]. This is based on Tenenbaum's treatment but is more explicit in the dependence on certain parameters, a feature that shall be crucial in our current application.

3.1. **Setup.** In what follows, we write complex numbers s as $s = \sigma + it$, where $\sigma := \mathrm{Re}(s)$ and $t := \mathrm{Im}(s)$. (This convention is relevant only for this section and the use of σ will not create any confusion with the notation for the sum-of-divisors function.) For a non-negative u, we use $\log^+ u$ to denote the quantity $\max\{0, \log u\}$ (the positive part of $\log u$), with the convention that $\log^+ 0 = 0$.

Given $\delta \in (0,1]$, a complex number z, and positive real numbers c_0 and M, we say that the Dirichlet series $F(s)$ has property $\mathcal{P}(z; c_0, \delta, M)$ if the function

$$G(s; z) := F(s)\zeta(s)^{-z}$$

satisfies the following two conditions:

(i) $G(s; z)$ continues analytically into the region $\sigma \ge 1 - c_0/(1 + \log^+|t|)$, and

(ii) $|G(s; z)| \le M(1 + |t|)^{1-\delta}$ for all s in this same region.

Given complex numbers z and w, along with positive numbers c_0, M and $\delta \in (0,1]$, we say that a Dirichlet series $F(s) := \sum_{n=1}^{\infty} a_n n^{-s}$ has type $\mathcal{T}(z, w; c_0, \delta, M)$ if:

(i) $F(s)$ has property $\mathcal{P}(z; c_0, \delta, M)$, and

(ii) there is a sequence $\{b_n\}_{n=1}^{\infty}$ of nonnegative real numbers satisfying $|a_n| \le b_n$ for all n, such that the Dirichlet series $\sum_{n=1}^{\infty} b_n n^{-s}$ has property $\mathcal{P}(w; c_0, \delta, M)$.

The following is the special case of Theorem A.13 in [6] with $N = 0$.

Proposition 3.1. *Fix $A \geq 1$ and let z, w be complex numbers satisfying $|z|, |w| \leq A$. Let c_0, δ, M be positive real numbers with $c_0 \leq 2/11$ and $\delta \leq 1$. Let $F(s) = \sum_{n=1}^{\infty} a_n/n^s$ be a Dirichlet series of type $\mathcal{T}(z, w; c_0, \delta, M)$. Then, uniformly for $x \geq \exp(8^{1/A} \max\{\delta^{-1-1/A}, 2/c_0\})$, we have*

$$\sum_{n \leq x} a_n = \frac{x}{(\log x)^{1-z}} \left(\frac{G(1;z)}{\Gamma(z)} + O(MR(x)) \right),$$

where

$$R(x) = \left(\frac{1}{\delta^{2A+3/2}} + \frac{1}{c_0^{2A+1}} \right) \exp\left(-\frac{1}{6} \sqrt{c_0 \delta \log x} \right) + \frac{1}{c_0 \log x}$$

and the implied constant depends at most on A.

Here we have corrected some typos in [6]; the expression for $R(x)$ there has an extra factor of M throughout as well as an extra factor of x in its first term.

3.2. **Proof of Lemma 2.1.** We claim that the Dirichlet series $F(s) := \sum_{n \geq 1} \mathbb{1}_{P^-(n) > Y}\, \beta^{\Omega(n)}/n^s$ is of type $\mathcal{T}(\beta, |\beta|; 1/\log Y, 1, C_0(\log Y)^{|\beta|})$ for some absolute constant $C_0 > 0$. Indeed, in the half plane $\sigma > 1$, we find that

$$G(s) := F(s)\zeta(s)^{-\beta} = \prod_{p \leq Y} \left(1 - \frac{1}{p^s}\right)^{\beta} \prod_{p > Y} \left(1 - \frac{1}{p^s}\right)^{\beta} \left(1 - \frac{\beta}{p^s}\right)^{-1}, \tag{13}$$

and in the same half plane

$$\log G(s) = \beta \sum_{p \leq Y} \log\left(1 - \frac{1}{p^s}\right) + \sum_{p > Y} \left\{ \beta \log\left(1 - \frac{1}{p^s}\right) - \log\left(1 - \frac{\beta}{p^s}\right) \right\}. \tag{14}$$

Now since $Y \geq e^{11/2}$, we see that if $\sigma \geq 1 - 1/\log Y$, then $\sigma \geq 9/11$, so that $|\beta/p^s| \leq 2^{-\sigma} \leq 0.57$. Consequently for all such s, the second sum in (14) is

$$\sum_{p > Y} \left\{ \beta\left(-\frac{1}{p^s} + O\left(\frac{1}{p^{2\sigma}}\right) \right) + \left(\frac{\beta}{p^s} + O\left(\frac{1}{p^{2\sigma}}\right) \right) \right\} \ll \sum_{p > Y} \frac{1}{p^{2\sigma}} \leq \sum_{p > Y} \frac{1}{p^{18/11}} \ll 1.$$

This shows that the second sum in (14) converges absolutely and uniformly in the half plane $\sigma \geq 1 - 1/\log Y$, thus defining a holomorphic function in the same region. Furthermore, for all s in this half plane, we have

$$\begin{aligned} \log|G(s)| \leq |\log G(s)| \leq |\beta| \sum_{p \leq Y} \frac{1}{p^{\sigma}} + O(1) &\leq |\beta| \sum_{p \leq Y} \frac{1}{p} \exp\left(\frac{\log p}{\log Y} \right) + O(1) \\ &= |\beta| \sum_{p \leq Y} \frac{1}{p} + O\left(1 + \frac{1}{\log Y} \sum_{p \leq Y} \frac{\log p}{p} \right) = |\beta| \log_2 Y + O(1). \end{aligned}$$

We deduce that $|G(s)| \leq C_0(\log Y)^{|\beta|}$ for all $\sigma \geq 1 - 1/\log Y$, which shows that $F(s)$ has property $\mathcal{P}(\beta; 1/\log Y, 1, C_0(\log Y)^{|\beta|})$. By invoking this very observation with $|\beta|$ in place of β, we find that the Dirichlet series $\sum_{n \geq 1} |\beta|^{\Omega(n)}/n^s$ has property $\mathcal{P}(|\beta|; 1/\log Y, 1, C_0(\log Y)^{|\beta|})$, which establishes our claim.

An application of Proposition 3.1 with $A = 1$ now shows that for all $X \geq Y^{16} = \exp(16 \log Y)$, we have

$$\sum_{n \leq X} \mathbb{1}_{P^-(n)>Y} \beta^{\Omega(n)} = \frac{X}{(\log X)^{1-\beta}} \left\{ \frac{G(1)}{\Gamma(\beta)} + O\left(\frac{(\log Y)^{1+|\beta|}}{\log X} + (\log Y)^{3+|\beta|} \exp\left(-\frac{1}{6} \sqrt{\frac{\log X}{\log Y}} \right) \right) \right\}.$$

For $Y \leq Z^{1/(18 \log_2 Z)^2}$, we have $2 \log_2 Y + \log_2 Z \leq 3 \log_2 Z \leq \frac{1}{6} \sqrt{\frac{\log Z}{\log Y}}$, which shows that

$$\sum_{n \leq X} \mathbb{1}_{P^-(n)>Y} \beta^{\Omega(n)} = \frac{X}{(\log X)^{1-\beta}} \left\{ \frac{G(1)}{\Gamma(\beta)} + O\left(\frac{(\log Y)^{1+|\beta|}}{\log Z} \right) \right\}. \tag{15}$$

Finally,

$$\sum_{p>Y} \left\{ \beta \log\left(1 - \frac{1}{p}\right) - \log\left(1 - \frac{\beta}{p}\right) \right\} \ll \sum_{p>Y} \frac{1}{p^2} \ll \frac{1}{Y \log Y},$$

and by the prime number theorem (with the usual de la Vallée Poussin error term), we have

$$\prod_{p \leq Y} \left(1 - \frac{1}{p}\right)^{\beta} = \frac{e^{-\gamma\beta}}{(\log Y)^{\beta}} \left(1 + O(\exp(-C_0 \sqrt{\log Y}))\right),$$

for some absolute constant $C_0 > 0$. This shows that

$$G(1) = \prod_{p \leq Y} \left(1 - \frac{1}{p}\right)^{\beta} \prod_{p>Y} \left(1 - \frac{1}{p}\right)^{\beta} \left(1 - \frac{\beta}{p}\right)^{-1} = \frac{e^{-\gamma\beta}}{(\log Y)^{\beta}} \left(1 + O(\exp(-C_0 \sqrt{\log Y}))\right)$$

which from (15) yields the first of the claimed estimates in Lemma 2.1. Since the Gamma function has no zeros in the complex plane, we have $|\Gamma(s)| \gg 1$ for all s in a fixed compact region. As a consequence, $\beta\Gamma(\beta) = \Gamma(1 + \beta) \gg 1$ for all $|\beta| \leq 1$, yielding the second assertion of the lemma.

4. Distribution of the sum-of-divisors to odd moduli: Proof of Theorem 1.2

In the rest of this section, we abbreviate $\alpha(q)$ to α. We shall make frequent use of the fact that $\alpha = \prod_{\ell | q}(1 - 1/(\ell - 1)) \gg \exp\left(-\sum_{\ell \leq q} 1/\ell\right) \gg 1/\log_2(3q)$ for all odd q. The following proposition allows us to give a rough estimate on the count of $n \leq x$ for which $\sigma(n)$ is coprime to q, uniformly in odd moduli $q \leq (\log x)^K$.

Proposition 4.1. *Fix $K > 0$ and a multiplicative function f for which there exists a non-constant polynomial $F \in \mathbb{Z}[T]$ satisfying $f(p) = F(p)$ for all primes p. If x is sufficiently large and $q \leq (\log x)^K$ with $\alpha_F(q) := \frac{1}{\varphi(q)} \#\{u \bmod q : (uF(u), q) = 1\} > 0$, then*

$$\#\{n \leq x : (f(n), q) = 1\} = \frac{x}{(\log x)^{1-\alpha_F(q)}} \exp(O((\log\log(3q))^{O(1)})). \tag{16}$$

This is Proposition 2.1 in [32]; more precise results appear in work of Scourfield [37, 38]. By the above proposition, we obtain

$$\#\{n \leq x : (\sigma(n), q) = 1\} = \frac{x}{(\log x)^{1-\alpha}} \exp(O((\log\log(3q))^{O(1)})), \tag{17}$$

uniformly in odd $q \le (\log x)^K$. Since $\alpha \gg 1/\log_2(3q)$, this shows that the second of the two assertions of Theorem 1.2 is an immediate consequence of the first, so we need to show the first assertion of the theorem. For this purpose, we shall also need the following estimate [32, Lemma 2.4] on the sum of reciprocals of the primes at which a given polynomial is coprime to a modulus q that varies in a wide range.

Lemma 4.2. *Let $F(T) \in \mathbb{Z}[T]$ be a fixed nonconstant polynomial. For each positive integer q and each real number $x \ge 3q$,*

$$\sum_{p \le x} \frac{\mathbb{1}_{\gcd(F(p),q)=1}}{p} = \alpha_F(q)\log_2 x + O((\log\log(3q))^{O(1)}),$$

where $\alpha_F(q)$ is as defined in Proposition 4.1.

To establish the first assertion of Theorem 1.2, we set $y := \exp((\log x)^{\epsilon/2})$ and $z := x^{1/\log_2 x}$. From the count of $n \le x$ having $\sigma(n) \equiv a \pmod q$, we first eliminate those that are either z-smooth or are divisible by the square of a prime exceeding y. By well-known results on smooth numbers (for instance [40, Theorem 5.13 and Corollary 5.19, Chapter III.5]), the total contribution of the former n is at most $\Psi(x,z) \ll x/(\log x)^{(1+o(1))\log_3 x}$. On the other hand, the total contribution of the latter n is $\ll x/y$. Both of these contributions are negligible in comparison to the error term in the statement of Theorem 1.2.

Among the surviving n, we now remove those that have $P_2(n) \le y$. Any such n can be written as $n = mP$, where $P = P(n) > z$, m is y-smooth and $\sigma(n) = \sigma(m)\sigma(P) = \sigma(m)(P+1)$. Since $\sigma(n) \equiv a \pmod q$, it follows that $\sigma(m)$ must be coprime to q; moreover, for each choice of m, this congruence forces $P \in (z, x/m]$ into at most one coprime residue class modulo q. Hence for each choice of m, there are $\ll x/\varphi(q)m\log(z/q) \ll x\log_2 x/\varphi(q)m\log x$ many possible choices of P, by the Brun-Titchmarsh theorem. Thus, the total contribution of the $n \le x$ having $P_2(n) \le y$ that survived the filtering in the previous paragraph is

$$\ll \frac{x\log_2 x}{\varphi(q)\log x}\sum_{m:\, P^+(m)\le y}\frac{\mathbb{1}_{(\sigma(m),q)=1}}{m} \ll \frac{x\log_2 x}{\varphi(q)\log x}\exp\left(\sum_{p\le y}\frac{\mathbb{1}_{(p+1,\,q)=1}}{p}\right)$$
$$\ll \frac{x\log_2 x}{\varphi(q)(\log x)^{1-\alpha\epsilon/2}}\exp((\log_2(3q))^{O(1)}) \ll \frac{x}{\varphi(q)(\log x)^{1-2\alpha\epsilon/3}}.$$

Here we have estimated the sum $\sum_{p\le y}\mathbb{1}_{(p+1,\,q)=1}/p$ using Lemma 4.2 (on the polynomial $F(T) := T+1$) and recalled that $\alpha \gg 1/\log_2(3q) \gg 1/\log_3 x$ for all odd $q \le (\log x)^K$. Collecting estimates, we have so far shown that

$$\sum_{\substack{n\le x\\ \sigma(n)\equiv a \pmod q}} 1 = \sum_{\substack{n\le x\\ P(n)>z,\, P_2(n)>y\\ p>y \implies p^2\nmid n}} \mathbb{1}_{\sigma(n)\equiv a \pmod q} + O\left(\frac{x}{\varphi(q)(\log x)^{1-2\alpha\epsilon/3}}\right)$$
$$= \sum_{\substack{n\le x\\ P(n)>z,\, P_2(n)>y\\ p>y \implies p^2\nmid n}} \frac{1}{\varphi(q)}\sum_{\chi \bmod q}\overline{\chi}(a)\chi(\sigma(n)) + O\left(\frac{x}{\varphi(q)(\log x)^{1-2\alpha\epsilon/3}}\right),$$

where in the second line above, we have used the orthogonality of the Dirichlet characters mod q to detect the congruence $\sigma(n) \equiv a \pmod q$. With $\chi_{0,q}$ denoting the principal character modulo q, we may thus isolate the contribution of $\chi_{0,q}$ to obtain

(18)
$$\sum_{\substack{n\le x\\ \sigma(n)\equiv a \ (\mathrm{mod}\ q)}} 1 = \frac{1}{\varphi(q)} \sum_{\substack{n\le x\\ (\sigma(n),q)=1}} 1 + \frac{1}{\varphi(q)} \sum_{\chi\ne\chi_{0,q} \bmod q} \overline{\chi}(a) \sum_{\substack{n\le x\\ P(n)>z,\, P_2(n)>y\\ p>y \implies p^2\nmid n}} \chi(\sigma(n)) + O\left(\frac{x}{\varphi(q)(\log x)^{1-2\alpha\epsilon/3}}\right).$$

The second outer sum on the right hand side above is over the nontrivial characters χ mod q, and we have adapted our previous arguments to observe that there are $O(x/(\log x)^{1-2\alpha\epsilon/3})$ many $n \le x$ satisfying $(\sigma(n), q) = 1$ but failing at least one of the three conditions below:

(i) $P(n) > z$,

(ii) $p > y \implies p^2 \nmid n$,

(iii) $P_2(n) > y$.

Indeed, if n satisfies conditions (i) and (ii) but fails (iii), then n is of the form mP, where $P = P(n) \in (z, x/m]$, $P(m) \le y$ and $\sigma(n) = \sigma(m)(P + 1)$. Consequently, $\sigma(m)$ must be coprime to q, and the number of P given m is $\ll x/m \log z \ll x \log_2 x/m \log x$. Summing this expression over all possible m yields the observed bound.

In order to estimate the inner sums of $\chi(\sigma(n))$ occurring in (18), we start by modifying some of our initial arguments in the proof of Theorem 1.1. Any n with $P(n) > z$, $P_2(n) > y$ and without any repeated prime factor exceeding y can be uniquely written in the form $mP_j \cdots P_1$ for some $j \ge 2$, where $P_1 = P(n) > z$ and $P(m) \le y < P_j < \cdots < P_1$. As such, we have

$$\sum_{\substack{n\le x\\ P(n)>z,\, P_2(n)>y\\ p>y \implies p^2\nmid n}} \chi(\sigma(n)) = \sum_{j\ge 2} \sum_{\substack{m\le x\\ P(m)\le y}} \chi(\sigma(m)) \sum_{\substack{P_1,\dots,P_j\\ P_j\cdots P_1\le x/m\\ P_1>z,\ y<P_j<\cdots<P_1}} \chi(P_1 + 1)\cdots\chi(P_j + 1).$$

Define $\rho_\chi := \frac{1}{\varphi(q)} \sum_{v \bmod q} \chi_{0,q}(v)\chi(v + 1)$. By the Siegel-Walfisz Theorem, we see that

$$\begin{aligned}\sum_{y<p\le Y} \chi(p + 1) &= \sum_{\substack{v \bmod q\\ (v,q)=1}} \chi(v + 1) \sum_{\substack{y<p\le Y\\ p\equiv v \ (\mathrm{mod}\ q)}} 1 \\ &= \sum_{\substack{v \bmod q\\ (v,q)=1}} \chi(v + 1) \left\{\frac{1}{\varphi(q)} \sum_{y<p\le Y} 1 + O\left(Y \exp(-C_0\sqrt{\log y})\right)\right\} \qquad (19)\\ &= \rho_\chi(\pi(Y) - \pi(y)) + O(\varphi(q)Y \exp(-C_0\sqrt{\log y})),\end{aligned}$$

where C_0 is a constant depending at most on K. As in the proof of Theorem 1.1, we successively remove $\chi(P_1 + 1), \dots, \chi(P_j + 1)$, with the input from (1) replaced by the above estimate. This leads us to

$$\sum_{\substack{n\le x\\ P(n)>z,\, P_2(n)>y\\ p>y \implies p^2\nmid n}} \chi(\sigma(n)) = \sum_{j\ge 2} \frac{(\rho_\chi)^j}{(j-1)!} \sum_{\substack{m\le x\\ P(m)\le y}} \chi(\sigma(m)) \sum_{\substack{P_1,\dots,P_j\\ P_1>z,\ P_j\cdots P_1\le x/m\\ P_2,\dots,P_j\in(y,P_1)\text{ distinct}}} 1 + O(x\exp(-C_1\sqrt{\log y})), \tag{20}$$

for some constant $C_1>0$ depending at most on K.

Now the main term in the display above is absolutely bounded by

(21)

$$|\rho_\chi|^2 \sum_{j\ge 2} \frac{|\rho_\chi|^{j-2}}{(j-1)!} \sum_{\substack{m\le x\\ P(m)\le y\\ (\sigma(m),q)=1}} \sum_{\substack{P_2,\dots,P_j\in(y,x)\\ P_2\cdots P_j\le x/mz}} \sum_{z<P_1\le x/mP_2\cdots P_j} 1$$
$$\ll \frac{|\rho_\chi|^2 x}{\log z} \sum_{j\ge 2} \frac{|\rho_\chi|^{j-2}}{(j-1)!} \sum_{\substack{m\le x\\ P(m)\le y\\ (\sigma(m),q)=1}} \frac{1}{m} \sum_{P_2,\dots,P_j\in(y,x)} \frac{1}{P_2\cdots P_j}$$
$$\ll |\rho_\chi|^2 \frac{x(\log_2 x)^2}{\log x} \sum_{j\ge 2} \frac{(|\rho_\chi|\log_2 x)^{j-2}}{(j-2)!} \sum_{\substack{m\le x\\ P(m)\le y\\ (\sigma(m),q)=1}} \frac{1}{m} \ll |\rho_\chi|^2 \frac{x(\log_2 x)^2}{(\log x)^{1-|\rho_\chi|}} \exp\Bigg(\sum_{\substack{p\le y\\ (p+1,q)=1}} \frac{1}{p}\Bigg).$$

Invoking Lemma 4.2 to estimate the last sum in the above display, we obtain

$$\sum_{\substack{n\le x\\ P(n)>z,\, P_2(n)>y\\ p>y \implies p^2\nmid n}} \chi(\sigma(n)) \ll |\rho_\chi|^2 \frac{x}{(\log x)^{1-|\rho_\chi|-2\alpha\epsilon/3}} + x\exp(-C_1\sqrt{\log y}). \tag{22}$$

In order to be able to make use of this bound, we need to estimate the $|\rho_\chi|$. To this end, given a nontrivial character χ mod q we let $\mathfrak{f}(\chi)$ denote its conductor, so that $\mathfrak{f}(\chi)\mid q$ and $\mathfrak{f}(\chi)>1$. We can write χ uniquely in the form $\prod_{\ell^e\| q}\chi_\ell$, with χ_ℓ denoting a character mod ℓ^e that is nontrivial precisely when $\ell\mid\mathfrak{f}(\chi)$. Note that $\varphi(q)\rho_\chi=\prod_{\ell^e\|q} S_{\chi,\ell}$, where for each prime power $\ell^e\parallel q$,

$$S_{\chi,\ell} \coloneqq \sum_{v \bmod \ell^e} \chi_{0,\ell}(v)\chi_\ell(v+1) = \sum_{\substack{v \bmod \ell^e\\ (v,\ell)=1}} \chi_\ell(v+1) = \sum_{u \bmod \ell^e} \chi_\ell(u) - \sum_{\substack{u \bmod \ell^e\\ u\equiv 1 \pmod{\ell}}} \chi_\ell(u). \tag{23}$$

Here $\chi_{0,\ell}$ denotes the principal character mod ℓ^e and we have noted that as v runs over all the coprime residues mod ℓ^e, the expression $v+1$ runs over all the residues mod ℓ^e except for those congruent to 1 mod ℓ. The first sum above is $\mathbb{1}_{\chi_\ell=\chi_{0,\ell}}\,\varphi(\ell^e)$. To evaluate the second sum, we consider a primitive root g mod ℓ^e (which exists as ℓ is odd), and observe that the residues $\{u \bmod \ell^e : u\equiv 1 \pmod{\ell}\}$ are a permutation of the residues $\{g^{(\ell-1)k} \bmod \ell^e : 0\le k<\ell^{e-1}\}$.

Hence

$$\sum_{\substack{u \bmod \ell^e \\ u\equiv 1 \pmod{\ell}}} \chi_\ell(u) = \sum_{0\le k<\ell^{e-1}} \chi_\ell(g^{\ell-1})^k = \mathbb{1}_{(\chi_\ell)^{\ell-1}=\chi_{0,\ell}}\ \ell^{e-1} = \mathbb{1}_{\mathfrak{f}(\chi_\ell)\mid \ell}\ \ell^{e-1}, \tag{24}$$

with $\mathfrak{f}(\chi_\ell)$ denoting the conductor of χ_ℓ. Putting these observations together, we find that

$$S_{\chi,\ell} = \mathbb{1}_{\chi_\ell=\chi_{0,\ell}}\ \varphi(\ell^e) - \mathbb{1}_{\mathfrak{f}(\chi_\ell)\mid\ell}\ \ell^{e-1} = \mathbb{1}_{\mathfrak{f}(\chi_\ell)\mid\ell}\ \ell^{e-1}\left(\mathbb{1}_{\ell\nmid\mathfrak{f}(\chi)}\ (\ell-1)-1\right),$$

leading to

$$\rho_\chi = \prod_{\ell^e\| q} \frac{S_{\chi,\ell}}{\varphi(\ell^e)} = \mathbb{1}_{\mathfrak{f}(\chi)\text{ squarefree}} \prod_{\ell^e\| q}\left(\mathbb{1}_{\ell\nmid\mathfrak{f}(\chi)} - \frac{1}{\ell-1}\right) = \mathbb{1}_{\mathfrak{f}(\chi)\text{ squarefree}} \frac{(-1)^{\omega(\mathfrak{f}(\chi))}\alpha}{\prod_{\ell\mid\mathfrak{f}(\chi)}(\ell-2)}. \tag{25}$$

If $3\mid q$, let ψ denote the unique character mod q induced by the nontrivial character mod 3. Then any nonprincipal character $\chi\neq\psi$ mod q for which $\rho_\chi\neq 0$ has conductor $\mathfrak{f}(\chi)$ divisible by a prime at least 5, so that $|\rho_\chi|\le\alpha/3$ by (25). Consequently, (22) yields for all such χ,

$$\sum_{\substack{n\le x \\ P(n)>z,\,P_2(n)>y \\ p>y\implies p^2\nmid n}} \chi(\sigma(n)) \ll |\rho_\chi|^2 \frac{x}{(\log x)^{1-(1/3+2\epsilon/3)\alpha}} + x\exp(-C_1\sqrt{\log y}). \tag{26}$$

Since there are exactly $\prod_{\ell\mid d}(\ell-2)$ primitive characters modulo any squarefree integer d, equation (25) yields

$$\sum_{\chi \bmod q} |\rho_\chi|^2 \le \alpha^2 \sum_{\substack{d\mid q \\ d\text{ squarefree}}} \frac{1}{\prod_{\ell\mid d}(\ell-2)^2} \sum_{\substack{\chi \bmod q \\ \mathfrak{f}(\chi)=d}} 1 \le \alpha^2 \sum_{\substack{d\mid q \\ d\text{ squarefree}}} \frac{1}{\prod_{\ell\mid d}(\ell-2)} \le \alpha^2 \prod_{\ell\mid q}\frac{\ell-1}{\ell-2} = \alpha.$$

(This may be compared with the bound placed on the averages $\frac{1}{\varphi(q)}\sum_{v \bmod q}\chi_{0,q}(v)\chi(v-1)$ in the proof of the analogous Theorem 1.3 in [30].)

Summing (26) over all nontrivial characters $\chi\neq\psi$ mod q, and plugging the resulting bound into (18), we obtain

(27)

$$\sum_{\substack{n\le x \\ \sigma(n)\equiv a \pmod q}} 1 = \frac{1}{\varphi(q)} \sum_{\substack{n\le x \\ (\sigma(n),q)=1}} 1 + \frac{\mathbb{1}_{3\mid q}\overline{\psi}(a)}{\varphi(q)} \sum_{\substack{n\le x \\ P(n)>z,\,P_2(n)>y \\ p>y\implies p^2\nmid n}} \psi(\sigma(n)) + O\left(\frac{x}{\varphi(q)(\log x)^{1-\alpha(1/3+\epsilon)}}\right)$$

$$= \frac{1}{\varphi(q)} \sum_{\substack{n\le x \\ (\sigma(n),q)=1}} 1 + \frac{\mathbb{1}_{3\mid q}\overline{\psi}(a)}{\varphi(q)} \sum_{n\le x} \psi(\sigma(n)) + O\left(\frac{x}{\varphi(q)(\log x)^{1-\alpha(1/3+\epsilon)}}\right).$$

Here in passing to the second line above, we have recalled our previous bound on the count of $n\le x$ having $\sigma(n)$ coprime to q but failing one of the conditions (i)–(iii) in the discussion following (18).

Note that the last equality in (27) already establishes the first assertion of the theorem for moduli q coprime to 6. To complete the proof of the theorem, it thus remains to only consider the odd moduli q divisible by 3 and deal with the sum of $\psi(\sigma(n))$ occurring in (27). This is where we can directly apply Theorem 1.1. Indeed, since $3\mid q$, we find that a prime p satisfies

$(p+1,\, q)=1$ only if $p=3$ or $p\equiv 1 \pmod 3$. Thus for all $p>3$, we have $\psi(\sigma(p))=\psi(p+1)=-\mathbb{1}_{(p+1,\,q)=1}$. A straightforward calculation (analogous to (19)) by means of the Siegel-Walfisz theorem shows that the multiplicative function $f(n):=\psi(\sigma(n))$ satisfies the hypothesis (1) with y,z as chosen in the beginning of the section and with $\varrho:=-\frac{1}{\varphi(q)}\sum_{\substack{v \bmod q\\ (v(v+1),q)=1}} 1=-\alpha$, $M:=\varphi(q)$, $\mathcal{E}(y):=\exp(-C_0\sqrt{\log y})$ (for some absolute constant $C_0>0$). Since by lemma (4.2), $\sum_{p\le y}|f(p)|/p=\alpha\log_2 y+O((\log_2(3q))^{O(1)})$, we deduce that

$$\sum_{n\le x}\psi(\sigma(n))\ll\frac{x(\log_2 x)^{1+\alpha}}{(\log x)^{1+\alpha(1-\epsilon)}}\exp((\log_2(3q))^{O(1)})\ll\frac{x}{\log x},$$

and substituting this into (27) completes the proof of the theorem.

Remark. To substantiate our comment (made after the paragraph following the statement of Theorem 1.4) on the suggested optimality of the exponent 1/3 in the error term of Theorem 1.2, note that by (25), we have $\rho_\chi=\alpha/3$ for $\mathfrak{f}(\chi)=15$ (in the case when $15\mid q$).

5. Technical Preparation for Theorems 1.3 and 1.4

In order to establish Theorems 1.3 and 1.4, we will need to study the averages

$$\eta_\chi:=\eta_\chi(q):=\frac{1}{\varphi(q)}\sum_{v \bmod q}\chi_{0,q}(v)\chi(v^2+v+1)$$

which will play the roles of the averages ρ_χ that came up in the previous section. The following proposition will provide the key information on these averages that will prove to be crucial in our arguments.

Proposition 5.1. *There exists a set $\mathcal{S}$ of eighteen (fixed) squarefree positive integers coprime to 6 such that for all sufficiently large integers q and all nonprincipal characters χ mod q, the following two properties hold true:*

(i) *If $\mathfrak{f}(\chi)\notin\mathcal{S}$, then $|\eta_\chi|\le\widetilde{\alpha}(q)/4$.*

(ii) *If $\mathfrak{f}(\chi)\in\mathcal{S}$, then $\mathrm{Re}(\eta_\chi)\le\widetilde{\alpha}(q)/4$.*

In particular, we have $\mathrm{Re}(\eta_\chi)\le\widetilde{\alpha}(q)/4$ for all nontrivial characters χ mod q, and $|\eta_\chi|\le\widetilde{\alpha}(q)/4$ for all but a bounded number of characters χ mod q.

The following character sum bound, a special case of [7, Theoem 1.1], will be useful to give a proof of Proposition 5.1.

Lemma 5.2. *Let ℓ be a prime at least 5. Then for any integer $e\ge 2$ and any primitive character χ mod ℓ^e, we have*

$$\left|\sum_{v \bmod \ell^e}\chi(v^2+v+1)\right|\le\ell^{e/2}.$$

Proof of Proposition 5.1. We start by factoring $\chi =: \prod_{\ell^e \| q} \chi_\ell$, where each χ_ℓ is as usual a character mod ℓ^e. This allows us to factor η_χ as $\prod_{\ell^e \| q} \eta_{\chi,\ell}$, where

$$\eta_{\chi,\ell} \coloneqq \frac{1}{\varphi(\ell^e)} \sum_{v \bmod \ell^e} \chi_{0,\ell}(v)\chi_\ell(v^2+v+1).$$

Since $\widetilde{\alpha}(\ell^e) = \frac{1}{\varphi(\ell^e)}\#\{v \bmod \ell^e : (v(v^2+v+1), \ell) = 1\}$, it is immediate that $|\eta_{\chi,\ell}| \le \widetilde{\alpha}(\ell^e) = \widetilde{\alpha}(\ell)$. Moreover, letting $e_\ell \coloneqq v_\ell(\mathfrak{f}(\chi))$ denote the exponent of the prime ℓ in the integer $\mathfrak{f}(\chi)$, we see that $\mathfrak{f}(\chi_\ell) = \ell^{e_\ell}$, so if $e_\ell = 0$ (i.e., $\ell \nmid \mathfrak{f}(\chi)$), then $\chi_\ell = \chi_{0,\ell}$ and $\eta_{\chi,\ell} = \widetilde{\alpha}(\ell)$.

Assume that χ_2 is nontrivial, so that $e_2 \ge 2$. Letting $U_{2^{e_2}}$ denote the multiplicative group mod 2^{e_2} we observe that the map $U_{2^{e_2}} \to U_{2^{e_2}} : v \mapsto v^2+v+1$, being injective, is also bijective. As a consequence

$$\eta_{\chi,2} = \frac{1}{\varphi(2^{e_2})} \sum_{\substack{v \bmod 2^{e_2} \\ (v,2)=1}} \chi_2(v^2+v+1) = \frac{1}{\varphi(2^{e_2})} \sum_{\substack{u \bmod 2^{e_2} \\ (u,2)=1}} \chi_2(u) = 0,$$

leading to $\eta_\chi = \eta_{\chi,2} \prod_{\ell | q:\ \ell>2} \eta_{\chi,\ell} = 0$. Hence in the rest of the argument, it suffices to consider only those characters χ mod q for which χ_2 is trivial, that is, for which $\mathfrak{f}(\chi)$ is not a multiple of 4. Since $3 \nmid q$ and χ is nontrivial mod q, it must then be the case that $\chi_\ell \ne \chi_{0,\ell}$ for some prime $\ell \ge 5$ dividing q.

Consider a prime $\ell \ge 5$ dividing q for which χ_ℓ is nontrivial. Letting χ_ℓ also denote the primitive chraracter mod ℓ^{e_ℓ} that induces χ_ℓ mod ℓ^e, we find that

$$\begin{aligned}
\eta_{\chi,\ell} &= \frac{1}{\varphi(\ell^e)} \cdot \ell^{e-e_\ell} \sum_{v \bmod \ell^{e_\ell}} \chi_{0,\ell}(v)\chi_\ell(v^2+v+1) = \frac{1}{\varphi(\ell^{e_\ell})} \sum_{v \bmod \ell^{e_\ell}} \chi_{0,\ell}(v)\chi_\ell(v^2+v+1) \\
&= \frac{1}{\varphi(\ell^{e_\ell})} \left\{ \sum_{v \bmod \ell^{e_\ell}} \chi_\ell(v^2+v+1) - \sum_{\substack{v \bmod \ell^{e_\ell} \\ v \equiv 0 \ (\mathrm{mod}\ \ell)}} \chi_\ell(v^2+v+1) \right\} \\
&= \frac{1}{\varphi(\ell^{e_\ell})} \left\{ \sum_{v \bmod \ell^{e_\ell}} \chi_\ell(v^2+v+1) - \mathbb{1}_{e_\ell=1} \right\}.
\end{aligned} \tag{28}$$

In the last equality above, we have observed that the map $\{v \bmod \ell^{e_\ell} : v \equiv 0 \pmod{\ell}\} \to \{u \bmod \ell^{e_\ell} : u \equiv 1 \pmod{\ell}\} : v \mapsto v^2+v+1$ being an injection from one set to another of the same cardinality is also a bijection. By (24), this led to

$$\sum_{\substack{v \bmod \ell^{e_\ell} \\ v \equiv 0 \ (\mathrm{mod}\ \ell)}} \chi_\ell(v^2+v+1) = \sum_{\substack{u \bmod \ell^{e_\ell} \\ u \equiv 1 \ (\mathrm{mod}\ \ell)}} \chi_\ell(u) = \mathbb{1}_{e_\ell=1}\ \ell^{e_\ell-1} = \mathbb{1}_{e_\ell=1}.$$

If $\ell \ge 5$ and $e_\ell \ge 2$, then by Lemma 5.2, we have

$$|\eta_{\chi,\ell}| = \frac{1}{\varphi(\ell^{e_\ell})} \left| \sum_{v \bmod \ell^{e_\ell}} \chi_\ell(v^2+v+1) \right| \le \frac{\ell^{e_\ell/2}}{\varphi(\ell^{e_\ell})} \le \frac{\ell^{1-e_\ell/2}}{\ell-1}. \tag{29}$$

Hence if $v_\ell(\mathfrak{f}(\chi)) = e_\ell \geq 2$ for some prime $\ell \geq 5$ dividing q, then

$$\frac{|\eta_\chi|}{\widetilde{\alpha}(q)} \leq \frac{|\eta_{\chi,\ell}|}{\widetilde{\alpha}(\ell)} \leq \frac{\ell^{1-e_\ell/2}}{\ell-1} \cdot \left(\frac{\ell-1}{\ell-3}\right)^{\mathbb{1}_{\ell \equiv 1 \ (\mathrm{mod}\ 3)}} \leq \frac{1}{4}.$$

This shows that $|\eta_\chi| \leq \widetilde{\alpha}(q)/4$ for all characters χ mod q whose conductor is not squarefree.

Consider now a nontrivial character χ mod q whose conductor is squarefree. Then by (28) and the Weil bounds (see for instance, [42, Corollary 2.3]), we find that

$$|\eta_{\chi,\ell}| = \frac{1}{\varphi(\ell)} \left| \sum_{v \bmod \ell} \chi_\ell(v^2+v+1) - 1 \right| \leq \frac{\ell^{1/2}+1}{\ell-1}. \tag{30}$$

Applying this bound for each prime $\ell \geq 5$ dividing q, we find that

$$\frac{|\eta_\chi|}{\widetilde{\alpha}(q)} \leq \prod_{\substack{\ell \mid q \\ \ell > 2}} \mu_\ell, \quad \text{where } \mu_\ell := \mathbb{1}_{\ell \equiv 1 \ (\mathrm{mod}\ 3)} \left(\frac{\ell^{1/2}+1}{\ell-3}\right) + \mathbb{1}_{\ell \geq 5,\ \ell \equiv 2 \ (\mathrm{mod}\ 3)} \left(\frac{\ell^{1/2}+1}{\ell-1}\right).$$

Note that $\mu_\ell \in (0,1)$ for all primes $\ell \geq 5$. Since the functions $(\ell^{1/2}+1)/(\ell-3)$ and $(\ell^{1/2}+1)/(\ell-1)$ are both strictly decreasing, we observe the following cases in which $|\eta_\chi|/\widetilde{\alpha}(q) \leq 1/4$. For $i \in \{1,2\}$, we let $\omega_i(r)$ denote the number of distinct primes dividing an integer r that are congruent to i mod 3.

- If $P^+(\mathfrak{f}(\chi)) \geq 29$, then $|\eta_\chi|/\widetilde{\alpha}(q) \leq \frac{29^{1/2}+1}{29-3} < 0.246$.
- If $\omega(\mathfrak{f}(\chi)) \geq 4$, then one of the following three possibilities must hold:
  (i) Either $\omega_1(\mathfrak{f}(\chi)) \geq 3$, in which case $|\eta_\chi|/\widetilde{\alpha} \leq \mu_7\mu_{13}\mu_{19} < 0.141$, **OR**
  (ii) $\omega_2(\mathfrak{f}(\chi)) \geq 3$, in which case $|\eta_\chi|/\widetilde{\alpha} \leq \mu_5\mu_{11}\mu_{17} < 0.112$, **OR**
  (iii) $\omega_1(\mathfrak{f}(\chi)) = \omega_2(\mathfrak{f}(\chi)) = 2$, in which case $|\eta_\chi|/\widetilde{\alpha} \leq \mu_7\mu_{13} \cdot \mu_5\mu_{11} < 0.147$.
- If $\omega(\mathfrak{f}(\chi)) = 3$ but $\mathfrak{f}(\chi)$ does not lie in the set $A_0 := \{5 \cdot 7 \cdot 11, 5 \cdot 7 \cdot 13\}$, then
  $$|\eta_\chi|/\widetilde{\alpha} \leq \max\{\mu_5\mu_{11}\mu_{13},\ \mu_5\mu_7\mu_{17},\ \mu_7\mu_{11}\mu_{13},\ \mu_5\mu_7\mu_{19}\} < 0.247.$$
- If $\omega(\mathfrak{f}(\chi)) = 2$ but $\mathfrak{f}(\chi)$ is not a member of the set $B_0 := \{7 \cdot 13, 7 \cdot 19\} \cup \{5 \cdot 11, 5 \cdot 17\} \cup \{5 \cdot 7, 5 \cdot 13, 5 \cdot 19, 7 \cdot 11, 7 \cdot 17\}$, then
  $$|\eta_\chi|/\widetilde{\alpha} \leq \max\{\mu_{13}\mu_{19},\ \mu_{11}\mu_{17},\ \mu_5\mu_{23},\ \mu_7\mu_{23},\ \mu_{11}\mu_{13}\} < 0.241.$$

Hence, defining $\mathcal{S}$ to be the set $\{5, 7, 11, 13, 17, 19, 23\} \cup A_0 \cup B_0$, we have shown that $|\eta_\chi| \leq \widetilde{\alpha}(q)/4$ for all characters χ mod q whose conductor $\mathfrak{f}(\chi)$ does not lie in the set $\mathcal{S}$. It thus only remains to show that for all χ mod q with $\mathfrak{f}(\chi) \in \mathcal{S}$, we have $\mathrm{Re}(\eta_\chi) \leq \widetilde{\alpha}(q)/4$. For such characters, the identity (28) shows that

$$\begin{aligned} \eta_\chi &= \prod_{\substack{\ell \mid q \\ \ell \nmid \mathfrak{f}(\chi)}} \widetilde{\alpha}(\ell) \cdot \prod_{\ell \mid \mathfrak{f}(\chi)} \left(\frac{1}{\varphi(\ell)} \sum_{\substack{v \bmod \ell \\ v \not\equiv 0 \bmod \ell}} \chi_\ell(v^2+v+1) \right) \\ &= \frac{\widetilde{\alpha}(q)}{\prod_{\substack{\ell \mid \mathfrak{f}(\chi) \\ \ell \equiv 1 \ (\mathrm{mod}\ 3)}} (\ell-3) \cdot \prod_{\substack{\ell \mid \mathfrak{f}(\chi) \\ \ell \equiv 2 \ (\mathrm{mod}\ 3)}} (\ell-1)} \sum_{\substack{v \bmod \mathfrak{f}(\chi) \\ \gcd(v, \mathfrak{f}(\chi))=1}} \psi(v^2+v+1), \end{aligned} \tag{31}$$

where $\psi := \prod_{\ell \mid \mathfrak{f}(\chi)} \chi_\ell$ denotes the primitive character mod $\mathfrak{f}(\chi)$ inducing χ. So we need only show that

(32)

$$\max_{Q \in \mathcal{S}} \frac{1}{\prod_{\substack{\ell \mid Q \\ \ell \equiv 1 \pmod 3}} (\ell - 3) \cdot \prod_{\substack{\ell \mid Q \\ \ell \equiv 2 \pmod 3}} (\ell - 1)} \max_{\substack{\psi \bmod Q \\ \psi \text{ primitive}}} \mathrm{Re}\left(\sum_{\substack{v \bmod Q \\ \gcd(v,Q)=1}} \psi(v^2+v+1) \right) \le \frac{1}{4}.$$

But $\mathcal{S}$ consists of only eighteen moduli, so this can be verified by a short Sage code. This completes the proof of Proposition 5.1. □

Remark. The aforementioned Sage code actually shows that equality is attained in (32) for $Q \in \{5, 7, 13, 35\} \subset \mathcal{S}$. In other words, for such Q, there exist primitive characters ψ mod Q for which

$$\frac{1}{\prod_{\substack{\ell \mid Q \\ \ell \equiv 1 \pmod 3}} (\ell - 3) \cdot \prod_{\substack{\ell \mid Q \\ \ell \equiv 2 \pmod 3}} (\ell - 1)} \cdot \mathrm{Re}\left(\sum_{\substack{v \bmod Q \\ \gcd(v,Q)=1}} \psi(v^2+v+1) \right) = \frac{1}{4}.$$

As we shall see in the proof of Proposition 5.4 below, the averages η_χ will play the roles of the parameter ϱ from Theorem 1.1. This supports our previous comment on the expected optimality of the "1/4" in the exponent of $\log x$ in the error terms of Theorems 1.3 and 1.4.

We shall also need the following analogue of Proposition 4.1, which gives a count for the main term in Theorems 1.3 and 1.4. In what follows, we abbreviate $\widetilde{\alpha}(q)$ to $\widetilde{\alpha}$. It will be important to keep in mind that $\widetilde{\alpha} = \prod_{\substack{\ell \mid q \\ \ell \equiv 1 \pmod 3}} (1 - 2/(\ell - 1)) \gg \exp\left(-2 \sum_{\substack{\ell \le q \\ \ell \equiv 1 \pmod 3}} 1/\ell \right) \gg 1/\log_2(3q)$.

Lemma 5.3. *Fix $K > 0$. We have*

$$\#\{n \le x : (\sigma(n), q) = 1\} = \frac{x^{1/2}}{(\log x)^{1-\widetilde{\alpha}}} \exp(O((\log\log(3q))^{O(1)})), \tag{33}$$

uniformly in even $q \le (\log x)^K$ such that $3 \nmid q$.

Proof. Our key observation is that since q is even, $\sigma(n)$ is coprime to q if and only if n is of the form $2^k m^2$ for some integer $k \ge 0$ and some odd integer m satisfying $\gcd(\sigma(m^2), q) = 1$; this follows from the fact that $\sigma(n) = \prod_{p^k \| n} \sigma(p^k) \equiv \prod_{p^k \| n:\ p>2} (k+1) \pmod 2$. In particular, if n is of the form r^2 for some integer r, then $\gcd(\sigma(n), q) = 1$. As such, the left hand side of (33) is no less than

$$\sum_{\substack{r \le x^{1/2} \\ (\sigma(r^2), q) = 1}} 1 \ge \frac{x^{1/2}}{(\log x)^{1-\widetilde{\alpha}}} \exp(O((\log\log(3q))^{O(1)})).$$

Here to write the last bound above, we have invoked Proposition 4.1 with $f(n) := \sigma(n^2)$ for which $F(T) := T^2 + T + 1$ and $\alpha_F(q) = \widetilde{\alpha}(q)$.

To obtain the upper bound, it suffices to show that the expression on the right hand side of (33) bounds (from above) the number of possible tuples (k, m) of non-negative integers for which m is odd, $2^k m^2 \le x$ and $\gcd(\sigma(m^2), q) = 1$. The contribution of those tuples for which $k > 20 \log_2 x / \log 2$ is no more than

$$\sum_{k > \frac{20 \log\log x}{\log 2}} \sum_{m \le \sqrt{x/2^k}} 1 \;\le\; x^{1/2} \sum_{k > \frac{20 \log_2 x}{\log 2}} \frac{1}{2^{k/2}} \;\ll\; \frac{x^{1/2}}{(\log x)^{10}}; \tag{34}$$

this is negligible compared to the right hand side of (33). On the other hand, if $k \le 20 \log_2 x/\log 2$, then $\sqrt{x/2^k} \ge x^{1/2}/(\log x)^{10} \ge x^{1/3}$. Consequently $q \le \left(\log \sqrt{x/2^k}\right)^{3K}$, and another application of Proposition 4.1 shows that given k, the number of possible m is at most

$$\sum_{\substack{m \le \sqrt{x/2^k} \\ (\sigma(m^2), q) = 1}} 1 \ll \frac{\sqrt{x/2^k}}{(\log \sqrt{x/2^k})^{1-\widetilde{\alpha}}} \exp\left(O\big((\log_2(3q))^{O(1)}\big)\right)$$

$$\ll \frac{x^{1/2}}{2^{k/2} (\log x)^{1-\widetilde{\alpha}}} \exp\left(O\big((\log_2(3q))^{O(1)}\big)\right).$$

Here we have noted that since $k \le 20 \log_2 x/\log x$, we have

$$\left(\log \sqrt{\frac{x}{2^k}}\right)^{1-\widetilde{\alpha}} = \left(\frac{1}{2}\log x\right)^{1-\widetilde{\alpha}} \left(1 + O\left(\frac{k}{\log x}\right)\right) = \left(\frac{1}{2}\log x\right)^{1-\widetilde{\alpha}} \left(1 + O\left(\frac{\log_2 x}{\log x}\right)\right).$$

Summing the bound in the above display over all $k \ge 0$ establishes the upper bound in (33). $\square$

To start proving Theorems 1.3 and 1.4, we set $y := \exp((\log x)^{\epsilon/4})$ and $z := x^{1/\log_2 x}$. We shall show that for even $q \le (\log x)^K$ and any coprime residue class a mod q, the dominant contribution to the count of $n \le x$ satisfying $\sigma(n) \equiv a \pmod{q}$ comes from those n which have sufficiently many large prime divisors. More precisely, these are the n which have at least six prime factors exceeding y counted with multiplicity.

Proposition 5.4. *Fix $K > 0$ and $\epsilon \in (0, 1)$. We have*

$$\sum_{\substack{n \le x:\ P_6(n) > y \\ \sigma(n) \equiv a \pmod{q}}} 1 = \frac{1}{\varphi(q)} \sum_{\substack{n \le x \\ (\sigma(n), q) = 1}} 1 + O\left(\frac{x^{1/2}}{\varphi(q)(\log x)^{1-\widetilde{\alpha}(q)(1/4+\epsilon)}}\right), \tag{35}$$

uniformly in coprime residue classes a mod q to moduli $q \le (\log x)^K$ satisfying $\gcd(q, 6) = 2$.

Proof. We start by bounding the contribution of the z-smooth n to the left hand side. By the observation made at the start of the proof of Lemma 5.3, any such n can be written in the form $2^k m^2$ for some $k \ge 0$ and some z-smooth odd m. The number of possibilities of k given m is $O(\log x)$. This in fact shows that

$$\sum_{\substack{n \le x:\ P(n) \le z \\ \gcd(\sigma(n), q) = 1}} 1 \le \sum_{\substack{m \le x^{1/2} \\ P(m) \le z}} \sum_{k \le \frac{\log x}{\log 2}} 1 \ll \log x \sum_{\substack{m \le x^{1/2} \\ P(m) \le z}} 1 \ll \frac{x^{1/2}}{(\log x)^{(1/2+o(1)) \log_3 x}}, \tag{36}$$

where in the last step we have again invoked [40, Theorem 5.13 and Corollary 5.19, Chapter III.5]. The last expression above is negligible in comparison to the error term in (35).

Next, we bound the contribution of those n which are divisible by the fourth power of a prime exceeding y. As before, any such n can be written in the form $2^k m^2$ for some $k \ge 0$ and some odd $m \le \sqrt{x/2^k}$ where (this time) m is divisible by the square of a prime exceeding y. Given k, the number of possibilities of m is no more than

$$\sum_{p>y} \sum_{\substack{m \le \sqrt{x/2^k} \\ p^2 | m}} 1 \le \sqrt{\frac{x}{2^k}} \sum_{p>y} \frac{1}{p^2} \ll \frac{\sqrt{x/2^k}}{y}.$$

Summing this bound over all $k \ge 0$, we find that

$$\sum_{\substack{n \le x:\ (\sigma(n),q)=1 \\ \exists p>y \text{ s.t. } p^4 | n}} 1 \ll \frac{\sqrt{x}}{y}, \tag{37}$$

which is also negligible compared to the error term in (35).

By (36) and (37), we may thus ignore the contribution of those n to the left hand side of (35), which are either z-smooth or are divisible by the fourth power of a prime exceeding y. In order to complete the proof of the proposition, it thus remains to show that

$$\sum_{\substack{n \le x:\ P_6(n)>y \\ P(n)>z;\ p>y \implies p^4 \nmid n \\ \sigma(n) \equiv a \pmod q}} 1 = \frac{1}{\varphi(q)} \sum_{\substack{n \le x \\ (\sigma(n),q)=1}} 1 + O\left(\frac{x^{1/2}}{\varphi(q)(\log x)^{1-\widetilde{\alpha}(q)(1/4+\epsilon)}}\right). \tag{38}$$

To prove this estimate, we start by invoking the orthogonality of the Dirichlet characters mod q to write

$$\sum_{\substack{n \le x:\ P_6(n)>y \\ P(n)>z;\ p>y \implies p^4 \nmid n \\ \sigma(n) \equiv a \pmod q}} 1 = \frac{1}{\varphi(q)} \sum_{\substack{n \le x:\ P_6(n)>y \\ P(n)>z;\ p>y \implies p^4 \nmid n \\ \gcd(\sigma(n),q)=1}} 1 + \frac{1}{\varphi(q)} \sum_{\chi \ne \chi_{0,q} \bmod q} \overline{\chi}(a) \sum_{\substack{n \le x:\ P_6(n)>y \\ P(n)>z;\ p>y \implies p^4 \nmid n}} \chi(\sigma(n)). \tag{39}$$

We remove the additional conditions in the first sum on the right hand side above. To begin with, we observe that up to a negligible error, we may ignore the condition $P_6(n) > y$: indeed, any $n \le x$ which violates this condition but satisfies all the other conditions in the first sum can be written in the form $2^k m^2$ where m is odd but not z-smooth, has no repeated prime factor exceeding y, and satisfies $P_6(m^2) \le y$ and $\gcd(\sigma(m^2), q) = 1$. As such, m can be written in the form rP for some prime $P > z$ and some odd r coprime to P which satisfies $P_2(r) \le y$ and $\gcd(\sigma(r^2), q) = 1$. Altogether, we find that

$$\sum_{\substack{n \le x:\ P_6(n) \le y \\ P(n)>z;\ p>y \implies p^4 \nmid n \\ \gcd(\sigma(n),q)=1}} 1 \le \sum_{k \ge 0} \sum_{\substack{r \le x:\ P_2(r) \le y \\ (\sigma(r^2),q)=1}} \sum_{z < P \le \sqrt{x/2^k r^2}} 1$$

$$\ll \frac{x^{1/2}}{\log z}\sum_{k\geq 0}\frac{1}{2^{k/2}}\sum_{\substack{r\leq x:\ P_2(r)\leq y\\ (\sigma(r^2),q)=1}}\frac{1}{r}\ll\frac{x^{1/2}\log_2 x}{\log x}\sum_{\substack{r\leq x:\ P_2(r)\leq y\\ (\sigma(r^2),q)=1}}\frac{1}{r}. \tag{40}$$

Any r counted in the above sum can be written in the form AB, where $P(B)\leq y<P(A)$, so that A is either 1 or a prime and $\sigma(r^2)=\sigma(A^2)\sigma(B^2)$. Given B, the sum of $1/A$ over all possible A is at most $1+\sum_{p\leq x}1/p\ll\log_2 x$. We infer that

$$\begin{aligned}\sum_{\substack{r\leq x:\ P_2(r)\leq y\\ (\sigma(r^2),q)=1}}\frac{1}{r}&\ll(\log_2 x)\sum_{\substack{B\leq x:\ P(B)\leq y\\ (\sigma(B^2),q)=1}}\frac{1}{B}\ll(\log_2 x)\exp\left(\sum_{p\leq y}\frac{\mathbb{1}_{(p^2+p+1,\,q)=1}}{p}\right)\\ &\ll(\log x)^{\widetilde{\alpha}\epsilon/4}(\log_2 x)\exp\left(O\big((\log_2(3q))^{O(1)}\big)\right),\end{aligned} \tag{41}$$

where in the last line above, we have invoked Lemma 4.2 on the polynomial $F(T)\coloneqq T^2+T+1$. Inserting the above bound into (40), we obtain

$$\sum_{\substack{n\leq x:\ P_6(n)\leq y\\ P(n)>z;\ p>y\implies p^4\nmid n\\ \gcd(\sigma(n),q)=1}}1\ll\frac{x^{1/2}(\log_2 x)^2}{(\log x)^{1-\widetilde{\alpha}\epsilon/4}}\exp\left(O\big((\log_2(3q))^{O(1)}\big)\right)\ll\frac{x^{1/2}}{(\log x)^{1-\widetilde{\alpha}\epsilon/2}}, \tag{42}$$

whereupon from (39), it follows that

$$\begin{aligned}\sum_{\substack{n\leq x:\ P_6(n)>y\\ P(n)>z;\ p>y\implies p^4\nmid n\\ \sigma(n)\equiv a\ (\mathrm{mod}\ q)}}1&=\frac{1}{\varphi(q)}\sum_{\substack{n\leq x\\ (\sigma(n),q)=1}}1\\ &+\frac{1}{\varphi(q)}\sum_{\chi\neq\chi_{0,q}\bmod q}\overline{\chi}(a)\sum_{\substack{n\leq x\\ P(n)>z,\ P_6(n)>y\\ p>y\implies p^4\nmid n}}\chi(\sigma(n))+O\left(\frac{x^{1/2}}{\varphi(q)(\log x)^{1-\widetilde{\alpha}\epsilon/2}}\right).\end{aligned} \tag{43}$$

Here we have also used (36) and (37) respectively to remove the conditions "$P(n)>z$" and "$p>y\implies p^4\nmid y$" occurring in the first sum on the right hand side of (39).

In order to estimate the inner sums on $\chi(\sigma(n))$ in (43), we proceed analogously to our proof of Theorem 1.2. For a given nontrivial character χ mod q, any n counted in the aforementioned sum can be uniquely written in the form $MP_1^2\cdots P_j^2$ for some $j\geq 3$, some y-smooth M and some primes $P_1,\dots,P_j$, which satisfy $P_1>z$ and $y<P_j<\cdots<P_2<P_1$. (Here the condition $j\geq 3$ is a consequence of $P_6(n)>y$.) Proceeding as in the proof of Theorem 1.1, and using the estimate

$$\sum_{y<p\leq Y}\chi(p^2+p+1)=\eta_\chi(\pi(Y)-\pi(y))+O(\varphi(q)Y\exp(-C_0\sqrt{\log y}))$$

in place of (19), we obtain the following analogue of (20)

$$\sum_{\substack{n\le x\\ P(n)>z,\ P_6(n)>y\\ p>y \implies p^4\nmid n}} \chi(\sigma(n)) = \sum_{j\ge 3} \frac{(\eta_\chi)^j}{(j-1)!} \sum_{\substack{M\le x\\ P(M)\le y}} \chi(\sigma(M)) \sum_{\substack{P_1,\dots,P_j\\ P_1>z,\ P_j\cdots P_1\le\sqrt{x/M}\\ P_2,\dots,P_j\in(y,P_1)\text{ distinct}}} 1 + O(x^{1/2}\exp(-C_1\sqrt{\log y})), \tag{44}$$

where $C_1 = C_1(K)$ is a constant. Bounding the main term above as in (21), we deduce that

$$\sum_{\substack{n\le x\\ P(n)>z,\ P_6(n)>y\\ p>y \implies p^4\nmid n}} \chi(\sigma(n)) \ll |\eta_\chi|^3 \frac{x^{1/2}(\log_2 x)^3}{(\log x)^{1-|\eta_\chi|}} \sum_{\substack{M\le x\\ P(M)\le y\\ (\sigma(M),q)=1}} \frac{1}{M^{1/2}} + x^{1/2}\exp(-C_1\sqrt{\log y}). \tag{45}$$

To estimate the sum on M above, we recall that, by the observation made at the start of the proof of Lemma 5.3, any M counted in the sum can be uniquely written in the form $2^k m^2$ for some $k \ge 0$ and some odd y-smooth m satisfying $\gcd(\sigma(m^2), q) = 1$. We thus obtain

$$\begin{aligned}\sum_{\substack{M\le x\\ P(M)\le y\\ (\sigma(M),q)=1}} \frac{1}{M^{1/2}} &\le \sum_{k\ge 0} \frac{1}{2^{k/2}} \sum_{\substack{m\le\sqrt{x/2^k}\\ P(m)\le y\\ (\sigma(m^2),q)=1}} \frac{1}{m}\\ &\ll \exp\left(\sum_{p\le y} \frac{\mathbb{1}_{(p^2+p+1,\,q)=1}}{p}\right) \ll (\log x)^{\widetilde{\alpha}\epsilon/4} \exp\left(O\left((\log_2(3q))^{O(1)}\right)\right).\end{aligned} \tag{46}$$

Inserting this bound into (45) yields

$$\sum_{\substack{n\le x\\ P(n)>z,\ P_6(n)>y\\ p>y \implies p^4\nmid n}} \chi(\sigma(n)) \ll |\eta_\chi|^3 \frac{x^{1/2}(\log_2 x)^3}{(\log x)^{1-|\eta_\chi|-\widetilde{\alpha}\epsilon/4}} \exp\left(O\left((\log_2(3q))^{O(1)}\right)\right) + x^{1/2}\exp(-C_1\sqrt{\log y}).$$

Assume that $\mathfrak{f}(\chi) \notin \mathcal{S}$, where $\mathcal{S}$ is the set of eighteen positive integers considered in Proposition 5.1. Then $|\eta_\chi| \le \widetilde{\alpha}/4$, and we obtain, for all such characters χ mod q,

$$\sum_{\substack{n\le x\\ P(n)>z,\ P_6(n)>y\\ p>y \implies p^4\nmid n}} \chi(\sigma(n)) \ll |\eta_\chi|^3 \frac{x^{1/2}}{(\log x)^{1-\widetilde{\alpha}(1/4+\epsilon/2)}} + x^{1/2}\exp(-C_1\sqrt{\log y}). \tag{47}$$

Now from the computations in (29) and (30), we see that for each nontrivial character χ mod q, we have (with $e_\ell := v_\ell(\mathfrak{f}(\chi))$ as before),

$$|\eta_\chi| = \prod_{\ell\mid q} |\eta_{\chi,\ell}| \le \prod_{\ell\mid\mathfrak{f}(\chi)} \frac{\ell^{e_\ell/2}+1}{\varphi(\ell^{e_\ell})} \le \prod_{\ell\mid\mathfrak{f}(\chi)} \ell^{-e_\ell/2}\left(1+O\left(\frac{1}{\ell^{1/2}}\right)\right) \le \mathfrak{f}(\chi)^{-1/2}\exp(O(\sqrt{\log q})).$$

Since there are no more than d characters mod q having conductor d, we obtain

$$\sum_{\chi\ne\chi_{0,q} \bmod q} |\eta_\chi|^3 \le \sum_{d\mid q} d\cdot\frac{1}{d^{3/2}}\exp(O(\sqrt{\log q})) \le \exp(O(\sqrt{\log q})),$$

where we have noted that $\sum_{d|q} d^{-1/2} \le \prod_{\ell|q}(1+O(\ell^{-1/2})) \le \exp(O(\sqrt{\log q}))$. Summing the bound (47) over all nonprincipal characters χ mod q having $\mathfrak{f}(\chi) \notin \mathcal{S}$, and invoking the bound on $\sum_{\chi \ne \chi_{0,q} \bmod q} |\eta_\chi|^3$ obtained above, we find that

$$\sum_{\substack{\chi \ne \chi_{0,q} \bmod q \\ \mathfrak{f}(\chi)\notin \mathcal{S}}} \left| \sum_{\substack{n\le x:\ P(n)>z \\ P_6(n)>y;\ p>y \implies p^4\nmid n}} \chi(\sigma(n)) \right| \ll \frac{x^{1/2}}{(\log x)^{1-\widetilde{\alpha}(1/4+\epsilon)}}. \tag{48}$$

It remains to consider the characters χ mod q whose conductors lie in the set $\mathcal{S}$. For each such character, we may invoke (42), (36) and (37) to obtain

$$\sum_{\substack{n\le x \\ P_6(n)>y,\ P(n)>z \\ p>y \implies p^4\nmid n}} \chi(\sigma(n)) = \sum_{n\le x} \chi(\sigma(n)) + O\left(\frac{x^{1/2}}{(\log x)^{1-\widetilde{\alpha}\epsilon/2}}\right).$$

Recalling the observation made at the start of the proof of Lemma 5.3 along with the bound (34), we obtain
(49)

$$\sum_{\substack{n\le x \\ P_6(n)>y,\ P(n)>z \\ p>y \implies p^4\nmid n}} \chi(\sigma(n)) = \sum_{k\le \frac{20\log\log x}{\log 2}} \chi(\sigma(2^k)) \sum_{m\le \sqrt{x/2^k}} \mathbb{1}_{2\nmid m}\ \chi(\sigma(m^2)) \ + \ O\left(\frac{x^{1/2}}{(\log x)^{1-\widetilde{\alpha}\epsilon/2}}\right).$$

Finally, we invoke Theorem 1.1 on the multiplicative function $m \mapsto \mathbb{1}_{2\nmid m}\ \chi(\sigma(m^2))$ to bound each of the inner sums in the above display. Noting that $\sqrt{x/2^k} \ge x^{1/2}/(\log x)^{10} > z$ and that η_χ plays the role of ϱ, we deduce that the sums on m in the above display are all

$$\ll \frac{\sqrt{x/2^k}}{(\log z)^{1-\mathrm{Re}(\eta_\chi)}} \left((\log y)^{\widetilde{\alpha}} + \frac{(\log y)^{1+\widetilde{\alpha}}}{\log z}\right) \exp\left(\sum_{p\le y} \frac{\mathbb{1}_{(p^2+p+1,\,q)=1}}{p}\right)$$
$$\ll \frac{x^{1/2}(\log_2 x)^2}{2^{k/2}(\log x)^{1-\widetilde{\alpha}/4-\widetilde{\alpha}\epsilon/2}} \exp\left(O\big((\log_2(3q))^{O(1)}\big)\right) \ll \frac{x^{1/2}}{2^{k/2}(\log x)^{1-\widetilde{\alpha}(1/4+\epsilon)}}.$$

In the last line above, we have utilized the second assertion of Proposition 5.1 (namely that $\mathrm{Re}(\eta_\chi) \le \widetilde{\alpha}/4$) in conjunction with Lemma 4.2. Summing the above bound over all $k \ge 0$ and inserting into (49), we obtain

$$\sum_{\substack{n\le x \\ P_6(n)>y,\ P(n)>z \\ p>y \implies p^4\nmid n}} \chi(\sigma(n)) \ll \frac{x^{1/2}}{(\log x)^{1-\widetilde{\alpha}(1/4+\epsilon)}}$$

for all characters χ mod q having $\mathfrak{f}(\chi) \in \mathcal{S}$. We use this bound for each of the $O(1)$ nontrivial characters χ mod q having $\mathfrak{f}(\chi) \in \mathcal{S}$ and use (48) to deal with the rest of the characters mod q. Inserting these bounds into (43), we obtain the desired estimate (38), which completes the proof of the proposition. ☐

Remark. The proofs of Theorem 1.2 and Proposition 5.4 substantiate the comments following Theorem 1.1. Note that $\varrho = -\alpha$ for the sum $\sum_{n\le x} \psi(\sigma(n))$ at the end of section 4, so a direct application of [30, Theorem 1.1] (or the methods used to prove it) would be unable to detect the negative sign of ϱ and would yield a bound on this sum which would have the same order of magnitude as the main term (by Lemma 5.3). A similar phenomenon takes place in the proof of Proposition 5.4 for the sums of $\chi(\sigma(n))$ for the characters χ having conductors in the set $\mathcal{S}$. On the other hand, we cannot apply this paper's Theorem 1.1 for all the nontrivial characters χ mod q, for if we did, then the terms with $(\log y)^{1+|\varrho|}$ in Theorem 1.1 would culminate into a large error term that would stand in the way of achieving uniformity in q up to (fixed) large powers of $\log x$.

6. Distribution of the sum-of-divisors function to general even moduli: Proof of Theorem 1.3

We continue with y and z as defined in the previous section. Observe that the right hand sides of (4) and (35) are equal up to a negligible error: indeed any n having $P_6(n) \le q$ also has $P_6(n) \le y$, so that by (36), (37) and (42), the contribution of all such n to the right hand side of (4) is absorbed in the error term. This observation also shows that the first assertion (4) of the theorem implies the second (5), by means of Lemma 5.3 and the fact that $\widetilde{\alpha}(q) \gg 1/\log_2(3q)$. By Proposition 5.4, it thus suffices to show that

$$\sum_{\substack{n\le x:\ q<P_6(n)\le y\\ P(n)>z;\ p>y \implies p^4\nmid n\\ \sigma(n)\equiv a \pmod q}} 1 \ll \frac{x^{1/2}}{\varphi(q)(\log x)^{1-\widetilde{\alpha}\epsilon/2}} \tag{50}$$

in order to complete the proof of Theorem 1.3.

We write the left hand side of (50) as $\Sigma_1+\Sigma_2$, where Σ_1 denotes the count of the n contributing to the sum in (50) which are divisible by the fourth power of a prime exceeding q. First consider the contribution of the n counted in Σ_1. As before, the coprimality of $\sigma(n)$ with q guarantees that we can write $n = 2^k m^2$ for some $k \ge 0$ and some odd m satisfying $P_6(m^2) \le y$. Since n is divisible by the fourth power of a prime exceeding q, it follows that the squarefull part of m (i.e., the largest squarefull divisor of m) is divisible by a prime exceeding q. Hence m can be written in the form rSP, with r, S, P being pairwise coprime and satisfying $P_2(r) \le y$, $P = P(n) > z$ and with $S > q^2$ being squarefull. Altogether, we find that

(51)

$$\Sigma_1 \le \sum_{\substack{n\le x:\ P_6(n)\le y\\ P(n)>z;\ p>y \implies p^4\nmid n\\ \exists p>q:\ p^4\mid n\\ \sigma(n)\equiv a \pmod q}} 1 \le \sum_{k\ge 0}\ \sum_{\substack{r\le x^{1/2}:\ P_2(r)\le y\\ p>y\implies p^2\nmid r\\ (\sigma(r^2),q)=1}}\ \sum_{\substack{S>q^2\\ S \text{ squarefull}}}\ \sum_{z<P\le x^{1/2}/2^{k/2}rS} 1$$
$$\ll \frac{x^{1/2}}{\log z}\sum_{k\ge 0}\frac{1}{2^{k/2}} \sum_{\substack{r\le x^{1/2}:\ P_2(r)\le y\\ (\sigma(r^2),q)=1}} \frac{1}{r} \sum_{\substack{S>q^2\\ S \text{ squarefull}}} \frac{1}{S} \ll \frac{x^{1/2}\log_2 x}{q\log x} \sum_{\substack{r\le x^{1/2}:\ P_2(r)\le y\\ (\sigma(r^2),q)=1}} \frac{1}{r} \ll \frac{x^{1/2}}{q(\log x)^{1-\widetilde{\alpha}\epsilon/2}},$$

where have used (41) and the standard bound $\sum_{S>q^2 \text{ squarefull}} 1/S \ll 1/q$. Since the last expression above is absorbed in the right hand side of (50), it remains to show that the same is true for the sum Σ_2.

Now any n counted in Σ_2 has $P_6(n) > q$ but is not divisible by the fourth power of a prime exceeding q. Invoking the observation at the start of the proof of Lemma 5.3, we find that any such n can be written in the form $2^k r^2 (P_1P_2P_3)^2$, where $k \ge 0$, $P_2(r) \le y$, and P_1, P_2, P_3 are primes satisfying $P_1 = P(n) > z$, $q < P_3 < P_2 < P_1$, and $\sigma(n) = \sigma(2^k)\sigma(r^2)\prod_{j=1}^3 (P_j^2 + P_j + 1)$. Given k and r, the congruence $\sigma(n) \equiv a \pmod q$ forces $(P_1, P_2, P_3) \bmod q \in \mathcal{V}_q(a\sigma(2^k r^2)^{-1})$, where u^{-1} denotes the multiplicative inverse of a coprime residue u mod q, and for any coprime residue w mod q, we have defined

$$\mathcal{V}_q(w) := \left\{ (v_1, v_2, v_3) \in U_q^3 : \prod_{j=1}^3 (v_j^2 + v_j + 1) \equiv w \pmod q \right\}.$$

(Recall that U_q denotes the group of units modulo q.) Given k, r and $(v_1, v_2, v_3) \in \mathcal{V}_q(a\sigma(2^k r^2)^{-1})$, we bound the number of possible choices of P_1, P_2, P_3 which satisfy $(P_1, P_2, P_3) \equiv (v_1, v_2, v_3)$ mod q. Given P_2 and P_3, the number of possible $P_1 \in (z, x^{1/2}/2^{k/2} r P_2 P_3]$ satisfying $P_1 \equiv v_1 \pmod q$ is, by the Brun-Titichmarsh inequality,

$$\ll \frac{x^{1/2}/2^{k/2} r P_2 P_3}{\varphi(q) \log(z/q)} \ll \frac{x^{1/2} \log_2 x}{\varphi(q) \log x} \cdot \frac{1}{2^{k/2} r P_2 P_3}.$$

We now sum this bound over all $P_2, P_3 \in (q, x]$ satisfying $P_2 \equiv v_2 \pmod q$ and $P_3 \equiv v_3 \pmod q$. By Brun-Titchmarsh and partial summation, we have

$$\sum_{\substack{q < p \le x \\ p \equiv v \pmod q}} \frac{1}{p} \ll \frac{\log_2 x}{\varphi(q)}.$$

Applying this to the sums on P_2 and P_3, we find that given k, r and $(v_1, v_2, v_3) \in \mathcal{V}_q(a\sigma(2^k r^2)^{-1})$, the number of possible P_1, P_2, P_3 satisfying $(P_1, P_2, P_3) \equiv (v_1, v_2, v_3)$ mod q is

$$\ll \frac{1}{\varphi(q)^3} \cdot \frac{x^{1/2} (\log_2 x)^3}{2^{k/2} r \log x}.$$

Now let $V_q := \max\{\#\mathcal{V}_q(w) : w \in U_q\}$. Summing the above bound over all $(v_1, v_2, v_3) \in \mathcal{V}_q(a\sigma(2^k r^2)^{-1})$, and subsequently over all k and r, we obtain

$$\Sigma_2 \ll \frac{V_q}{\varphi(q)^3} \cdot \frac{x^{1/2} (\log_2 x)^4}{(\log x)^{1 - \widetilde{\alpha}\epsilon/4}} \exp\left(O\left((\log_2 (3q))^{O(1)} \right) \right). \tag{52}$$

To bound V_q, we consider an arbitrary coprime residue w mod q, and note that $\#\mathcal{V}_q(w) = \prod_{\ell^e \| q} \#\mathcal{V}_{\ell^e}(w)$ by the Chinese Remainder Theorem. Moreover, by orthogonality,

$$\begin{aligned}
\#\mathcal{V}_{\ell^e}(w) &= \sum_{v_1, v_2, v_3 \bmod \ell^e} \chi_{0,\ell}(v_1 v_2 v_3) \cdot \frac{1}{\varphi(\ell^e)} \sum_{\chi \bmod \ell^e} \overline{\chi}(w) \chi\left(\prod_{j=1}^3 (v_j^2 + v_j + 1) \right) \\
&= \frac{(\widetilde{\alpha}(\ell)\varphi(\ell^e))^3}{\varphi(\ell^e)} \left\{ 1 + \frac{1}{(\widetilde{\alpha}(\ell)\varphi(\ell^e))^3} \sum_{\chi \ne \chi_{0,\ell} \bmod \ell^e} \overline{\chi}(w) \left(\sum_{v \bmod \ell^e} \chi_{0,\ell}(v) \chi(v^2 + v + 1) \right)^3 \right\}.
\end{aligned}$$

Given $\chi \ne \chi_{0,\ell}$ mod ℓ^e, let ℓ^{e_0} denote the conductor of χ, so that $e_0 \in \{1, \dots, \ell^e\}$. Then with χ also denoting the primitive character mod ℓ^{e_0} inducing χ, the computations and arguments in (28), (29) and (30) reveal that if $\ell \geq 5$, then

$$\left|\sum_{v \bmod \ell^e} \chi_{0,\ell}(v)\chi(v^2+v+1)\right| = \ell^{e-e_0}\left|\sum_{v \bmod \ell^{e_0}} \chi(v^2+v+1) - \mathbb{1}_{e_0=1}\right| \ll \ell^{e-e_0}\cdot \ell^{e_0/2} \ll \ell^{e-e_0/2}.$$

Since there are at most $\varphi(\ell^{e_0})$ characters mod ℓ^e with conductor ℓ^{e_0}, we obtain

$$\#\mathcal{V}_{\ell^e}(w) = \frac{(\widetilde{\alpha}(\ell)\varphi(\ell^e))^3}{\varphi(\ell^e)}\left\{1+O\left(\frac{1}{\varphi(\ell^e)^3}\sum_{1\leq e_0\leq e}\varphi(\ell^{e_0})\ (\ell^{e-e_0/2})^3\right)\right\} \leq \frac{\varphi(\ell^e)^3}{\varphi(\ell^e)}\left\{1+O\left(\frac{1}{\ell^{1/2}}\right)\right\},$$

where we have recalled that for each odd prime ℓ dividing q, we have $\widetilde{\alpha}(\ell) \geq 1-2/(\ell-1) \geq 1/2$. Letting $e_1 := v_2(q)$ and multiplying the above bound over all the odd primes dividing q, we obtain

$$\frac{\#\mathcal{V}_q(w)}{\varphi(q)^3} \leq \frac{\#\mathcal{V}_{2^{e_1}}(w)}{\varphi(2^{e_1})^3}\prod_{\substack{\ell^e\|q\\ \ell>2}}\frac{1}{\varphi(\ell^e)}\left(1+O\left(\frac{1}{\ell^{1/2}}\right)\right) \leq \frac{\#\mathcal{V}_{2^{e_1}}(w)}{\varphi(2^{e_1})^2}\cdot\frac{1}{\varphi(q)}\exp(O(\sqrt{\log q}))$$

uniformly in coprime residues w mod q. Recalling (from the proof of Proposition 5.1) that the map $v \mapsto v^2+v+1$ is a bijection on the multiplicative group mod 2^{e_1}, we see that

$$\#\mathcal{V}_{2^{e_1}}(w) = \sum_{\substack{a_1,a_2,a_3 \bmod 2^{e_1}\\ a_1a_2a_3\equiv w \pmod{2^{e_1}}}} 1 = \sum_{\substack{a_1,a_2 \bmod 2^{e_1}\\ \gcd(a_1a_2,2)=1}} 1 = \varphi(2^{e_1})^2, \tag{53}$$

leading to the bound

$$\frac{\#\mathcal{V}_q(w)}{\varphi(q)^3} \leq \frac{1}{\varphi(q)}\exp(O(\sqrt{\log q}))$$

uniformly in coprime residues w mod q. Hence this bound also holds true for the ratio $V_q/\varphi(q)^3$, and inserting this latter bound into (52) we obtain

$$\Sigma_2 \ll \frac{x^{1/2}(\log_2 x)^4}{\varphi(q)(\log x)^{1-\widetilde{\alpha}\epsilon/4}}\exp(O(\sqrt{\log q})) \ll \frac{x^{1/2}}{\varphi(q)(\log x)^{1-\widetilde{\alpha}\epsilon/2}},$$

thus showing that Σ_2 is also absorbed into the right hand side of (50) and completing the proof of the theorem. □

6.1. **Optimality in the restriction $P_6(n) > q$.** We construct a counterexample demonstrating that the restriction $P_6(n) > q$ is optimal, in the sense that uniformity in $q \leq (\log x)^K$ fails if this restriction is weakened, or in other words, if the set of inputs n is slightly enlarged to those having fewer than 6 prime factors exceeding q. To do this, we define

$$\widetilde{\mathcal{V}}_q(w) := \left\{(v_1,v_2)\in U_q\times U_q:\ (v_1^2+v_1+1)(v_2^2+v_2+1)\equiv w \pmod q\right\}. \tag{54}$$

We shall first establish that

$$\#\widetilde{\mathcal{V}}_{\ell^2}(9\cdot 16^{-1}) \geq 2\ell^2\left(1+O\left(\frac{1}{\sqrt{\ell}}\right)\right) \tag{55}$$

uniformly in primes $\ell \geq 5$, with 16^{-1} denoting the multiplicative inverse of 16 mod ℓ. For each $(v_1,v_2)\in\widetilde{\mathcal{V}}_{\ell^2}(9\cdot16^{-1})$, we set $a_i \equiv v_i^2+v_i+1 \pmod{\ell^2}$, which is equivalent to $(2v_i+1)^2 \equiv 4a_i-3$

(mod ℓ^2). As such, we may write

$$\#\widetilde{\mathcal{V}}_{\ell^2}(9\cdot 16^{-1}) = \sum_{\substack{(a_1,a_2)\in U_{\ell^2}\times U_{\ell^2}\\ a_1a_2\equiv 9\cdot 16^{-1} \pmod{\ell^2}\\ \text{each } 4a_i-3 \text{ is a square mod } \ell^2}} \sum_{\substack{(v_1,v_2)\in U_{\ell^2}\times U_{\ell^2}\\ \text{each } (2v_i+1)^2\equiv 4a_i-3 \pmod{\ell^2}}} 1 \geq S_1+S_2, \tag{56}$$

where S_1 denotes the contribution of the case $4a_1-3\equiv 4a_2-3\equiv 0 \pmod{\ell^2}$ and S_2 denotes the contribution of the case $\ell\nmid(4a_1-3)(4a_2-3)$.

First of all, we see that

$$S_1 = \sum_{\substack{(v_1,v_2)\in U_{\ell^2}\times U_{\ell^2}\\ \text{each } (2v_i+1)^2\equiv 0 \pmod{\ell^2}}} 1 = \sum_{\substack{(v_1,v_2)\in U_{\ell^2}\times U_{\ell^2}\\ \text{each } v_i\equiv -2^{-1} \pmod{\ell}}} 1 = \ell^2. \tag{57}$$

We seek to put a lower bound on the sum S_2. To do this, we first note that the condition $\ell\nmid(4a_1-3)(4a_2-3)$ in conjunction with the condition that $4a_1-3$ and $4a_2-3$ are both squares mod ℓ^2 are together equivalent to the condition that $\left(\frac{4a_1-3}{\ell}\right)=\left(\frac{4a_2-3}{\ell}\right)=1$; indeed the forward direction is tautological, while the reverse implication is a consequence of Hensel's Lemma. In fact by the same lemma, we see that for each choice of $a_i\in U_\ell^2$ satisfying $\left(\frac{4a_i-3}{\ell}\right)=1$, the congruence $t^2\equiv 4a_i-3 \pmod{\ell^2}$ has exactly two distinct solutions t mod ℓ^2. If $a_i\not\equiv 1 \pmod{\ell}$, then $4a_i-3\not\equiv 1 \pmod{\ell}$, so that $t\not\equiv 1 \pmod{\ell}$ for both of the two aforementioned solutions, and each of them leads to a unique solution $v_i\in U_{\ell^2}$ (given by $2v_i+1\equiv t \pmod{\ell^2}$). Summarizing our argument, we have shown that

$$S_2 \geq \sum_{\substack{(a_1,a_2)\in U_{\ell^2}\times U_{\ell^2}\\ a_1a_2\equiv 9\cdot 16^{-1} \pmod{\ell^2}\\ \left(\frac{4a_1-3}{\ell}\right)=\left(\frac{4a_2-3}{\ell}\right)=1}} \sum_{\substack{(v_1,v_2)\in U_{\ell^2}\times U_{\ell^2}\\ \text{each } (2v_i+1)^2\equiv 4a_i-3 \pmod{\ell^2}}} 1 \geq 4 \sum_{\substack{(a_1,a_2)\in U_{\ell^2}\times U_{\ell^2}\\ 16a_1a_2\equiv 9 \pmod{\ell^2}\\ \text{each } a_i\not\equiv 1 \pmod{\ell},\ \left(\frac{4a_i-3}{\ell}\right)=1}} 1.$$

Now the condition $a_1a_2\equiv 9\cdot 16^{-1} \pmod{\ell^2}$ shows that if $a_1\equiv 1 \pmod{\ell}$, then $a_2\equiv 9\cdot 16^{-1} \pmod{\ell}$ which can be lifted to a residue class mod ℓ^2 in at most ℓ ways. This shows that ignoring the condition "each $a_i\not\equiv 1 \pmod{\ell}$" in the last sum in the above display incurs an error of $O(\ell)$. We deduce that

$$S_2 \geq 4 \sum_{\substack{(a_1,a_2)\in U_{\ell^2}\times U_{\ell^2}\\ 16a_1a_2\equiv 9 \pmod{\ell^2}\\ \left(\frac{4a_1-3}{\ell}\right)=\left(\frac{4a_2-3}{\ell}\right)=1}} 1 + O(\ell).$$

Moreover, for any $a_i\in U_{\ell^2}$ satisfying $\left(\frac{4a_i-3}{\ell}\right)=1$, we can write $4a_i-3$ in the form $u_i^2+\ell c_i \pmod{\ell^2}$ for some $u_i,c_i\in\{0,1,\ldots,\ell-1\}$ such that $\gcd(u_i,\ell)=1$. In fact, given a_i, there are exactly two possible choices of u_i and exactly one possible choice of c_i (this is because u_i can only be one of the two square roots of $4a_i-3$ mod ℓ, and either of them determines the same value of c_i via the congruence $c_i\equiv\frac{4a_i-3-u_i^2}{\ell}$ mod ℓ). Hence

$$S_2 \geq \sum_{\substack{(u_1,u_2)\in U_\ell\times U_\ell\\ (u_1^2+3)(u_2^2+3)\equiv 9 \pmod{\ell}}} \sum_{\substack{(c_1,c_2)\in\mathbb{Z}/\ell\mathbb{Z}\times\mathbb{Z}/\ell\mathbb{Z}\\ c_1(u_2^2+3)+c_2(u_1^2+3)\equiv\frac{9-(u_1^2+3)(u_2^2+3)}{\ell} \pmod{\ell}}} 1 + O(\ell), \tag{58}$$

where we have noted that since $4a_i \equiv u_i^2+3+\ell c_i \pmod{\ell^2}$, the condition $(u_1^2+3+\ell c_1)(u_2^2+3+\ell c_2) \equiv 16a_1a_2 \equiv 9 \pmod{\ell^2}$ can be rewritten as $\ell(c_1(u_2^2+3)+c_2(u_1^2+3)) \equiv 9-(u_1^2+3)(u_2^2+3) \pmod{\ell^2}$. Now given c_1, the congruence involving c_1 and c_2 in (58) determines c_2 uniquely mod ℓ. Varying c_1 over the ℓ possibilities, we thus find that

(59) $$S_2 \geq \ell \sum_{\substack{(u_1,u_2)\in U_\ell\times U_\ell \\ (u_1^2+3)(u_2^2+3)\equiv 9 \pmod{\ell}}} 1 + O(\ell) = \ell \sum_{\substack{(u_1,u_2)\in \mathbb{F}_\ell\times \mathbb{F}_\ell \\ (u_1^2+3)(u_2^2+3)=9 \text{ in } \mathbb{F}_\ell}} 1 + O(\ell),$$

where in the last equality above, we have noted that there is exactly one possible tuple $(u_1,u_2) \in \mathbb{F}_\ell \times \mathbb{F}_\ell$ satisfying $(u_1^2+3)(u_2^2+3) = 9$ in $\mathbb{F}_\ell$, in which either u_1 or u_2 is zero (namely the tuple $(u_1,u_2)=(0,0)$).

In order to estimate the last sum in (59), we proceed in a manner similar to the proof of Theorem 1.4(b) in [32]: we first show that the polynomial $G(X,Y) := (X^2+3)(Y^2+3)-9$ is absolutely irreducible over $\mathbb{F}_\ell[X,Y]$. [2] Indeed, assume that $G = UV$ for some $U, V \in \overline{\mathbb{F}}_\ell[X,Y]$; we wish to show that one of U or V must be constant. If either U or V is a polynomial only in Y (say $U(X,Y)=u(Y)$), then taking X to be a root $\theta \in \overline{\mathbb{F}}_\ell$ of the polynomial X^2+3 on both sides of the identity $G = UV$, we obtain $-9 = G(\theta, Y) = u(Y)V(\theta,Y)$ in the ring $\overline{\mathbb{F}}_\ell[Y]$, showing that $U(X,Y)=u(Y)$ must be constant. On the other hand, if neither U nor V is a polynomial in Y only, then by comparing the degrees in the variable X on both sides of the identity $H = UV$, we find that $U(X,Y) = u_1(Y)X + u_0(Y)$ and $V(X,Y) = v_1(Y)X + v_0(Y)$ for some $u_i, v_i \in \overline{\mathbb{F}}_\ell[Y]$. Comparing the coefficients of X on both sides of the identity

$$(Y^2+3)(X^2+3)-9 = (u_1(Y)X+u_0(Y))(v_1(Y)X+v_0(Y)),$$

we get the three identities $u_1(Y)v_1(Y) = Y^2+3$, $u_1(Y)v_0(Y)+u_0(Y)v_1(Y) = 0$ and $u_0(Y)v_0(Y) = 3Y^2$. Again, letting $\theta \in \overline{\mathbb{F}}_\ell$ be a root of the polynomial Y^2+3, the first of the three identities shows that $Y-\theta$ divides exactly one of u_1 or v_1 (as the polynomial Y^2+3 is separable over $\mathbb{F}_\ell$). Assuming without loss of generality that $(Y-\theta) \mid u_1(Y)$ (so that $\gcd(Y-\theta, v_1(Y))=1$), the second of the aforementioned identities forces $Y-\theta$ to divide $u_0(Y)$, leading to a contradiction in the third identity (since $3\theta^2 = -9 \neq 0$ in $\overline{\mathbb{F}}_\ell$). This establishes that G is indeed absolutely irreducible over $\mathbb{F}_\ell[X,Y]$.

Consequently, the variant of the Weil bound established in [23, Corollary 2(b)] yields, from (59),

$$S_2 \geq \ell(\ell + O(\sqrt{\ell})) + O(\ell) = \ell^2\left(1+O\left(\frac{1}{\sqrt{\ell}}\right)\right).$$

Combining this with (57) and (56) completes the proof of (55).

Now set $q := 2\left(\prod_{5\leq \ell\leq Y}\ell\right)^2$, where $Y \ll \log_2 x$ is a parameter to be chosen appropriately later. Then $q \leq (\log x)^{O(1)}$ and letting w_q denote the unique coprime residue mod q satisfying $w_q \equiv 9\cdot 16^{-1} \pmod{\ell^2}$ for all primes $\ell \in [5,Y]$, the lower bound (55) yields

(60)
$$\#\widetilde{\mathcal{V}}_q(w_q) = \prod_{5\leq\ell\leq Y}\#\widetilde{\mathcal{V}}_{\ell^2}(9\cdot 16^{-1}) \gg 2^{\pi(Y)}q\prod_{5\leq\ell\leq Y}\left(1+O\left(\frac{1}{\sqrt{\ell}}\right)\right) \gg 2^{\pi(Y)}q\exp(-C_2\sqrt{\log_2 x}).$$

[2]While this claim follows from the absolute irreducibility established in the proof of [32, Theorem 1.4(b)], we shall give a more straightforward self-contained argument that suffices for our current setting.

for some absolute constant $C_2 > 0$. Now consider any integer n of the form $P_1^2 P_2^2$, where P_1 and P_2 are primes satisfying $x^{1/10} < P_2 \le x^{1/6} < P_1 \le x^{1/2}/P_2$ and $(P_1, P_2) \bmod q \in \widetilde{\mathcal{V}}_q(w_q)$. Then $n \le x$, $P_4(n) = P_2 > x^{1/10} > q$ and $\sigma(n) = (P_1^2 + P_1 + 1)(P_2^2 + P_2 + 1) \equiv w_q \pmod q$. By the Siegel-Walfisz Theorem and partial summation, we obtain

$$\sum_{\substack{n \le x:\ P_4(n) > q \\ \sigma(n) \equiv w_q \ (\mathrm{mod}\ q)}} 1 \ge \sum_{(v_1, v_2) \in \widetilde{\mathcal{V}}_q(w_q)} \sum_{\substack{x^{1/10} < P_2 \le x^{1/6} \\ P_2 \equiv v_2 \ (\mathrm{mod}\ q)}} \sum_{\substack{x^{1/6} < P_1 \le x^{1/2}/P_2 \\ P_1 \equiv v_1 \ (\mathrm{mod}\ q)}} 1$$

$$\gg \sum_{(v_1, v_2) \in \widetilde{\mathcal{V}}_q(w_q)} \sum_{\substack{x^{1/10} < P_2 \le x^{1/6} \\ P_2 \equiv v_2 \ (\mathrm{mod}\ q)}} \frac{x^{1/2}/P_2}{\varphi(q) \log x} \gg \frac{\#\widetilde{\mathcal{V}}_q(w_q)}{\varphi(q)^2} \cdot \frac{x^{1/2}}{\log x}.$$

An application of (60) now yields

$$\sum_{\substack{n \le x:\ P_4(n) > q \\ \sigma(n) \equiv w_q \ (\mathrm{mod}\ q)}} 1 \gg \frac{2^{\pi(Y)}}{\varphi(q)} \cdot \frac{x^{1/2}}{\log x} \exp(-C_2 \sqrt{\log_2 x})$$

for some constant $C_2 > 0$. By Lemma 5.3, the quantity on the right hand side above grows strictly faster than the expected main term $\frac{1}{\varphi(q)} \sum_{\substack{n \le x \\ (\sigma(n), q) = 1}} 1$ as soon as $2^{\pi(Y)} > (\log x)^{(1+\delta)\widetilde{\alpha}}$ for some fixed $\delta > 0$, which in turn is equivalent to $\pi(Y) > (1+\delta)\widetilde{\alpha} \log_2 x / \log 2$. But now $\pi(Y) > Y/2 \log Y$, while

$$\widetilde{\alpha}(q) = \prod_{\substack{5 \le \ell \le Y \\ \ell \equiv 1 \ (\mathrm{mod}\ 3)}} \left(1 - \frac{2}{\ell - 1}\right) = \exp\left(-2 \sum_{\substack{5 \le \ell \le Y \\ \ell \equiv 1 \ (\mathrm{mod}\ 3)}} \frac{1}{\ell} + O(1)\right) < \frac{K_2}{\log Y} \tag{61}$$

for some absolute constant $K_2 > 0$, where we have used the prime number theorem in arithmetic progressions to estimate the sum on ℓ. As such, the desired condition $\pi(Y) > (1+\delta)\widetilde{\alpha} \log_2 x / \log 2$ holds as soon as $Y > 2K_2(1+\delta) \log_2 x / \log 2$, which is compatible with the only other condition $Y \ll \log_2 x$ needed on the parameter Y. Choosing Y accordingly, we have therefore established that the condition $P_6(n) > q$ in Theorem 1.3 cannot be weakened to $P_4(n) > q$ in the range of uniformity in q. Since the largest odd divisor of n is a perfect square, it follows that the restriction $P_6(n) > q$ in Theorem 1.3 is indeed optimal.

7. Distribution of the sum-of-divisors function to squarefree even moduli: Proof of Theorem 1.4

As in the beginning of the previous section, we can show that the right hand sides of (6) and (35) are equal up to a negligible error and that the first assertion (6) of the theorem implies the second (7). Indeed, by previous arguments, this only needs the following analogue (and immediate consequence) of the bound (42):

$$\sum_{\substack{n \le x:\ P_4(n) \le y \\ P(n) > z;\ p > y \implies p^4 \nmid n \\ \gcd(\sigma(n), q) = 1}} 1 \ll \frac{x^{1/2}}{(\log x)^{1 - \widetilde{\alpha}\epsilon/2}}.$$

Hence to complete the proof of the theorem, it suffices to show the following analogue of (50) uniformly in squarefree even moduli $q \le (\log x)^K$ and in coprime residues a mod q:

$$\sum_{\substack{n\le x:\ q<P_4(n)\le y\\ P(n)>z;\ p>y \implies p^4\nmid n\\ \sigma(n)\equiv a \pmod q}} 1 \ll \frac{x^{1/2}}{\varphi(q)(\log x)^{1-\widetilde{\alpha}\epsilon/2}}. \tag{62}$$

As before, we write the sum on the left hand side in the form $\widetilde{\Sigma}_1 + \widetilde{\Sigma}_2$, where $\widetilde{\Sigma}_1$ denotes the contribution of the n which are divisible by the fourth power of a prime exceeding q. Then with Σ_1 as defined in the previous section, it follows by the intermediate bounds in (51) that

$$\widetilde{\Sigma}_1 \le \sum_{\substack{n\le x:\ P_4(n)\le y\\ P(n)>z;\ p>y \implies p^4\nmid n\\ \exists p>q:\ p^4\mid n\\ \sigma(n)\equiv a \pmod q}} 1 \le \sum_{\substack{n\le x:\ P_6(n)\le y\\ P(n)>z;\ p>y \implies p^4\nmid n\\ \exists p>q:\ p^4\mid n\\ \sigma(n)\equiv a \pmod q}} 1 \ll \frac{x^{1/2}}{q(\log x)^{1-\widetilde{\alpha}\epsilon/2}}$$

is absorbed in the right hand side of (62).

In order to estimate the sum $\widetilde{\Sigma}_2$, we note that any n counted in this sum has $P_4(n) > q$ but is not divisible by the fourth power of a prime exceeding q. Consequently, as in the previous section, we may write $n = 2^k r^2 (P_1P_2)^2$ where $k \ge 0$, $P_2(r) \le y$, and P_1, P_2 are primes satisfying $P_1 = P(n) > z$, $q < P_2 < P_1$, and $\sigma(n) = \sigma(2^k)\sigma(r^2)\prod_{j=1}^2 (P_j^2 + P_j + 1)$. Given k and r, the congruence $\sigma(n) \equiv a \pmod q$ forces (P_1, P_2) mod $q \in \widetilde{\mathcal{V}}_q(a\sigma(2^k r^2)^{-1})$, where $\widetilde{\mathcal{V}}_q(w)$ is as defined in (54) for any coprime residue w mod q. Hereafter, setting $\widetilde{V}_q := \max\{\#\widetilde{\mathcal{V}}_q(w) :\ w \in U_q\}$ and replicating the arguments leading to (52) gives the following analogue of the latter bound:

$$\widetilde{\Sigma}_2 \ll \frac{\widetilde{V}_q}{\varphi(q)^2} \cdot \frac{x^{1/2}(\log_2 x)^3}{(\log x)^{1-\widetilde{\alpha}\epsilon/4}} \exp\big(O\big((\log_2(3q))^{O(1)}\big)\big). \tag{63}$$

Now since q is squarefree, we may write $\#\widetilde{\mathcal{V}}_q(w) = \prod_{\ell \mid q} \#\widetilde{\mathcal{V}}_\ell(w)$ for any coprime residue w mod q. Here $\#\widetilde{\mathcal{V}}_\ell(w)$ is no more than the number of $\mathbb{F}_\ell$-rational points of the polynomial $H(X,Y) := (X^2+X+1)(Y^2+Y+1) - w$. We claim that this latter number is no more than $\varphi(\ell)(1 + O(\ell^{-1/2}))$. By a computation similar to (53), this is true for $\ell = 2$ (without the multiplicative error term), so we may consider the case $\ell \ge 5$. But in fact, an argument entirely analogous to that given for the polynomial $(X^2+3)(Y^2+3) - 9$ in subsection 6.1, shows that the polynomial H is absolutely irreducible over $\mathbb{F}_\ell[X,Y]$. (Here again, it is important that $w \ne 0 \in \overline{\mathbb{F}}_\ell$ and that since $\ell \ge 5$, the polynomial $Y^2 + Y + 1$ is separable over $\mathbb{F}_\ell$.) As such, [23, Corollary 2(b)] establishes our claim.

As a consequence, we obtain $\#\widetilde{\mathcal{V}}_q(w) \le \prod_{\ell\mid q} \varphi(\ell)(1 + O(\ell^{-1/2})) \le \varphi(q)\exp(O(\sqrt{\log q}))$ uniformly in coprime residues w mod q. The same bound thus continues to hold for $\widetilde{V}_q$, and (63) shows that $\widetilde{\Sigma}_2$ is also absorbed in the right hand side of (62), establishing Theorem 1.4. □

7.1. **Optimality in the restriction** $P_4(n) > q$**.** The restriction $P_4(n) > q$ is crucial and optimal in the sense that weak equidistribution fails (in the range of uniformity in q provided by the theorem) as soon as one enlarges the set of inputs n to those having fewer prime factors exceeding q. Indeed, let $q := 2\prod_{5\le \ell \le Y} \ell$ for some parameter $Y \ll \log_2 x$ to be chosen appropriately. For a prime $P \in (q, x^{1/2}]$, the congruence $\sigma(P^2) \equiv 3 \pmod q$ holds for P lying in exactly $2^{\omega(q)-1}$ distinct coprime residue classes modulo q (namely those lying in the residue classes -2 or 1 modulo each of the odd prime divisors of q). As such, by the Siegel-Walfiz theorem, there are $\gg \frac{2^{\omega(q)}}{\varphi(q)} \frac{x^{1/2}}{\log x}$ many integers $n \le x$ having $P_2(n) > q$ and $\sigma(n) \equiv 3 \pmod q$, coming only from the squares of the primes lying in the interval $(x^{1/4}, x^{1/2}]$. By (33), the coprime residue class 3 mod q will be over-represented as soon as $2^{\omega(q)} > (\log x)^{(1+\delta)\widetilde{\alpha}(q)}$ for a fixed $\delta > 0$. By the same computation as in (61), we have $\widetilde{\alpha}(q) \ll 1/\log Y$, whereas $\omega(q) \ge Y/2\log Y$. The inequality $2^{\omega(q)} > (\log x)^{(1+\delta)\widetilde{\alpha}(q)}$ is thus ensured as soon as we choose $Y > K_1 \log_2 x$ for some appropriate constant $K_1 > 0$, a condition that is consistent with the only requirement $Y \ll \log_2 x$ on Y. This shows that the restriction $P_2(n) > q$ is inadequate to get weak equidistribution to moduli varying up to a fixed arbitrary power of $\log x$. Since n is of the form $2^k m^2$ for some odd m, it follows that the restriction $P_4(n) > q$ in Theorem 1.4 is optimal.

Remark. The above example may be compared with the one given in the discussion following the statement of Theorem 1.3 in [32].

Acknowledgements

This work was done in partial fulfillment of my PhD at the University of Georgia. As such, I would like to thank my advisor, Prof. Paul Pollack, for fruitful discussions and valuable comments. I would also like to thank the Department of Mathematics at UGA for their support and hospitality.

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