# The Landau-Selberg-Delange method for Dirichlet $L$-functions, and applications

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**Charles University Number Theory Seminar**

25 March, 2026

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$\rightsquigarrow$ Tauberian theory, mean values of multiplicative functions.

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**6.** Distribn. of invariant factors and elementary divisors of unit groups.

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Let $\chi$ be a Dirichlet character mod $q$. $L(s,\chi) := \sum_{n\geq 1} \chi(n)/n^s$.

$\chi : \mathbb{Z} \to \mathbb{C}$ is periodic with period $q$ and satisfies

- $\chi(a) = 0 \iff \gcd(a,q) > 1$, and
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- For any $\{a_n\}_n \subset \mathbb{C}$, $\exists\, b \in \mathbb{R} \cup \{\infty\}$ s.t. $F(s) = \sum_{n \geq 1} a_n/n^s$ is holomorphic for $\mathrm{Re}(s) > b$, while $\sum_{n \geq 1} a_n/n^s$ diverges for $\mathrm{Re}(s) < b$.

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**Behavior of $F(s)$ on $\{\mathrm{Re}(s) = b\}$?** All bets are off!

Eg: $\sum_{n\geq 1} \frac{1}{n}$ diverges, but $\sum_{n\geq 2} \frac{(-1)^n \cdot n/\log^2 n}{n^s}$ converges on $\mathrm{Re}(s) = 1$.

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(i) $F(s)$ converges absolutely on $\{\mathrm{Re}(s) > 1\}$, and

(ii) There exist $\alpha > 0$ and a function $G(s)$ holomorphic in a neighborhood of $s = 1$ with $G(1) \neq 0$, s. t. $F(s) = G(s)/(s-1)^{\alpha}$ on this neighborhood.

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Then $$\sum_{n\leq x} a_n \sim \frac{c_0\, x}{(\log x)^{1-\alpha}} \text{ as } x \to \infty,$$ for some constant $c_0 \in \mathbb{C}$.

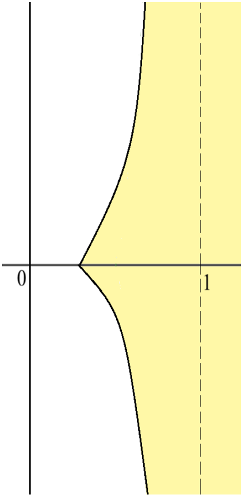
# The Classical Landau-Selberg-Delange "LSD" Setting

**Basic set-up:** Often it happens that

$$\sum_{n\geq 1} \frac{a_n}{n^s} = \zeta(s)^\alpha G(s) \text{ for all } s \text{ with } \mathrm{Re}(s) > 1,$$

where $\alpha \in \mathbb{C}$, and $G(s)$ is "well-behaved".

$G(s)$: analytic & bounded as a "nice" function of $s$, on a region like...



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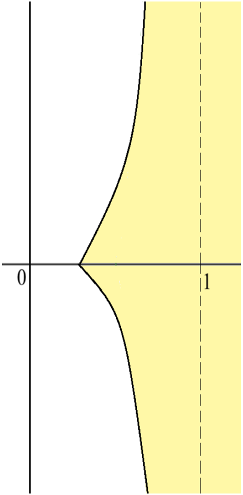


**Why this region?** $\zeta(s)$ can be continued meromorphically into a nonvanishing function on this region.

- Only simple pole at $s = 1$.

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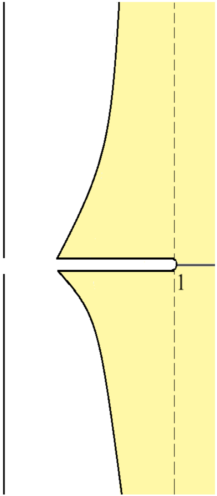
  **Note:** Possible branch point at $s = 1$.



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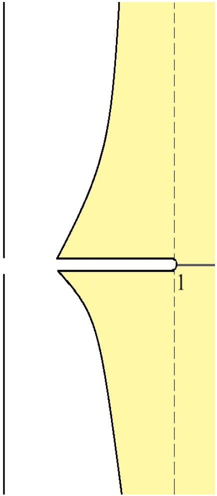
**Intuition: Why this assumption?**

- When $\{a_n\}_n$ is **multiplicative** (i.e. $a_{mn} = a_m\, a_n$ for $\gcd(m, n) = 1$), problems on $\{a_n\}_n$ reduce to $\{a_p\}_p$.



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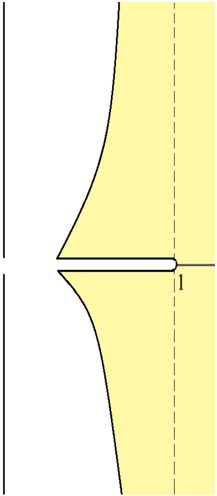

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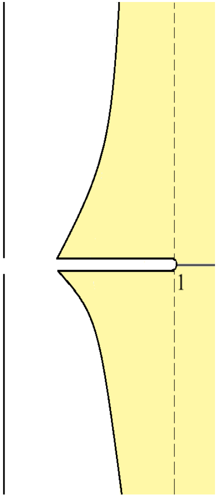
**Eg:** $a_n = \mathbb{1}(p \mid n \implies p \equiv 1 \bmod 4)$
$\implies a_p = \mathbb{1}_{p \equiv 1 \bmod 4} \implies \alpha = 1/2$.



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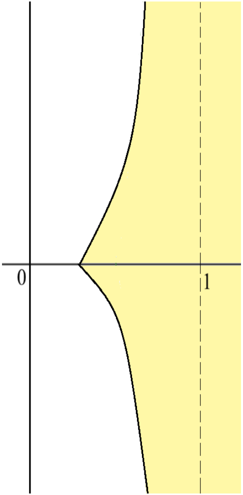

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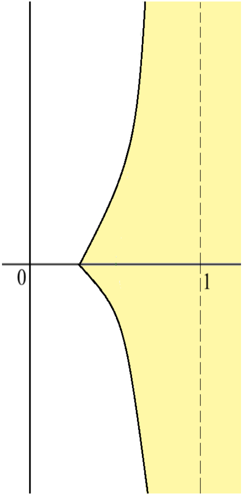

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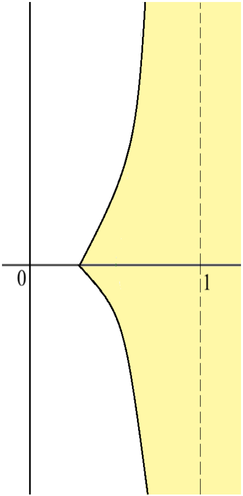
Delange-Ikehara $\Longrightarrow \sum_{n\leq x} a_n \sim \frac{c_0 \, x}{(\log x)^{1-\alpha}}$.



## The Classical Landau-Selberg-Delange "LSD" Method

**Theorem ("The LSD Method", Tenenbaum).** Assume that $\sum_{n\geq 1} a_n/n^s = \zeta(s)^\alpha G(s)$ for all $s$ with $\mathrm{Re}(s) > 1$, where $\alpha \in \mathbb{C}$, and $G(s)$ is "well-behaved". Then for some $c_0, \ldots, c_N \in \mathbb{C}$,

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**uniformly** in $x \geq 3$ and $N \geq 0$.

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**uniformly** in $x \geq 3$ and $N \geq 0$.

- **Uniformly:** implied constants independent of $x, N$.
- **Err:** Depends on how "well-behaved" $G$ is.
    - Does **NOT** always give the desired saving!

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For $\mathrm{Re}(s) > 1$, we have

$$\sum_{n \ge 1} \frac{\mathbb{1}_{n=\square+\square}}{n^s} = \sum_{m,a,b} \frac{1}{2^{ms}\, a^{2s}\, b^s}$$

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$$\sum_{n \ge 1} \frac{\mathbb{1}_{n=\Box+\Box}}{n^s} = \sum_{m,a,b} \frac{1}{2^{ms}\, a^{2s}\, b^s} = \left(\sum_{m\ge 0} \frac{1}{2^{ms}}\right) \left(\sum_{a} \frac{1}{a^{2s}}\right) \left(\sum_{b} \frac{1}{b^s}\right)$$

## An Example Application: Sum of squares

**Estimate** $\#\{n \le x \,:\, n = A^2 + B^2$ for some $A, B \in \mathbb{Z}\}$.

Every $n = \square + \square$ is uniquely of the form $2^m a^2 b$, where $m \ge 0$, where $a$ is composed of primes $\equiv 3 \bmod 4$, and $b$ is composed of primes $\equiv 1 \bmod 4$.

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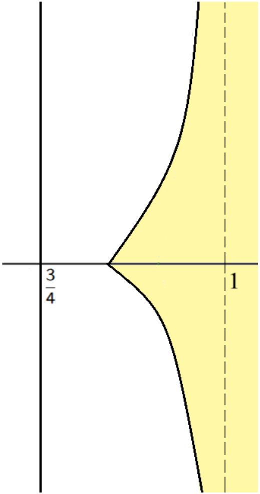
$$= \left(1 - \frac{1}{2^s}\right)^{-1} \cdot \prod_{\ell \equiv 3 \bmod 4} \left(1 + \frac{1}{\ell^{2s}} + \frac{1}{\ell^{4s}} + \dots\right)$$

$$\cdot \prod_{p \equiv 1 \bmod 4} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right)$$

Typical term of last product $= 1/(p_1^{e_1} \cdots p_k^{e_k})^s$ with $p_1 < \cdots < p_k$ primes $\equiv 1 \bmod 4$, and $e_1, \dots, e_k \in \mathbb{Z}^+$. Every $1/b^s$ appears exactly once.

$$\sum_{n\geq 1} \frac{\mathbb{1}_{n=\square+\square}}{n^s}$$

$$= \left(1-\frac{1}{2^s}\right)^{-1} \cdot \left(\prod_{\ell\equiv 3 \bmod 4} \left(1-\frac{1}{\ell^{2s}}\right)^{-1}\right) \cdot \left(\prod_{p\equiv 1 \bmod 4} \left(1-\frac{1}{p^s}\right)^{-1}\right)$$



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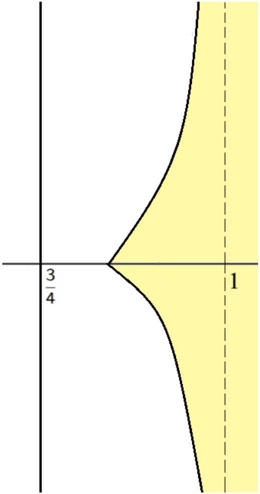

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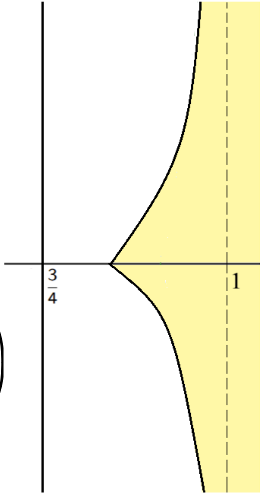

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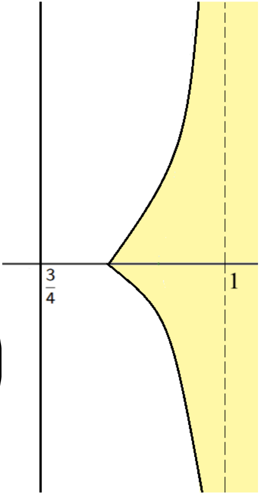

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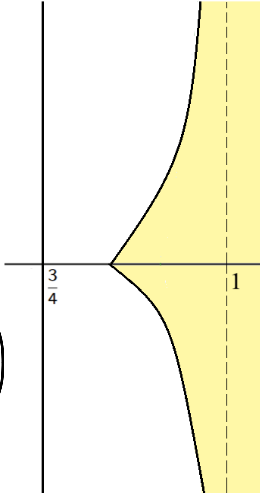


$$\#\{n\leq x : n=\square+\square\} = \sum_{n\leq x} \mathbb{1}_{n=\square+\square}$$

$$= \frac{c_0\, x}{(\log x)^{1/2}} + \frac{c_1\, x}{(\log x)^{3/2}} + \cdots + \frac{c_N\, x}{(\log x)^{N+1/2}} + O\left(\frac{x}{(\log x)^{(N+1)+1/2}}\right).$$

# Extending the LSD theorem...

**Tenenbaum:** Assume $\sum_{n\geq 1} a_n/n^s = \zeta(s)^\alpha G(s)$ for $\mathrm{Re}(s) > 1$, where $\alpha \in \mathbb{C}$, and $G(s)$ is well-behaved. Then uniformly in $x \geq 3$, $N \geq 0$,

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(*Recall:* $L(s,\chi) = \sum_{n\geq 1} \chi(n)/n^s$, and there are $\varphi(q)$ many $L(s,\chi)$'s.)

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**Eg.:** Let $a_n = \mathbb{1}(p \mid n \implies p \equiv a \bmod q)$. Then $\alpha_\chi = \overline{\chi}(a)/\varphi(q)$.

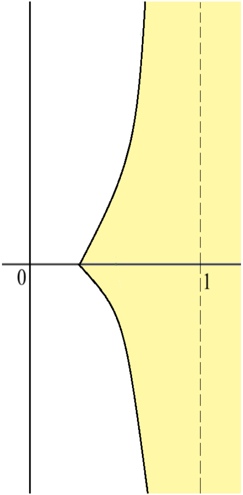
## Not as simple as it might seem...

**Facts**

**(i)** With $\chi_0(n) := \mathbb{1}_{(n,q)=1}$, we have

$L(s, \chi_0) \approx \zeta(s)$ simple pole at $s = 1$.

**(ii)** For all $\chi \neq \chi_0$ mod $q$, $L(s, \chi)$ analytically continues to $\mathbb{C}$.



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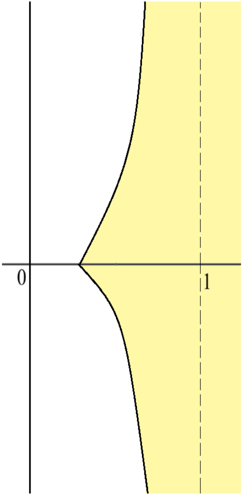
What if we write $\sum_{n\geq 1} a_n/n^s = \left(\prod_{\chi \bmod q} L(s,\chi)^{\alpha_\chi}\right) \cdot G(s) \approx \zeta(s)^{\alpha_{\chi_0}} \cdot H(s)$?



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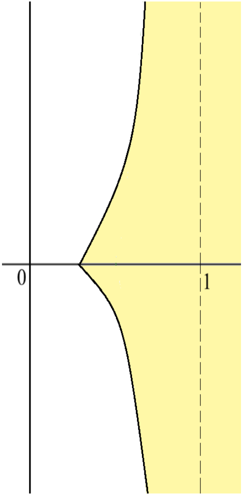
$H(s) := G(s) \cdot \prod_{\chi \neq \chi_0 \bmod q} L(s, \chi)^{\alpha_\chi}$
needs to be well-behaved.

analytic + suitably bounded as a function of $s$ in a region like...



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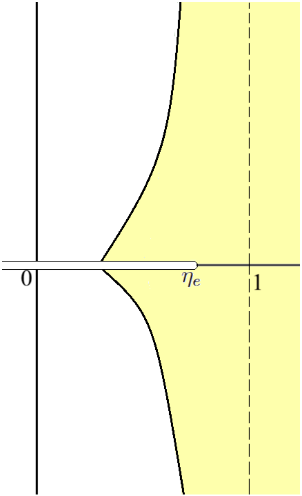

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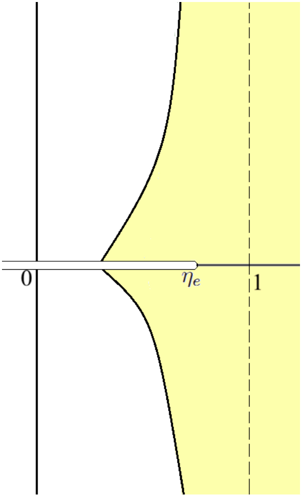
**Issue 2:** Best known direct bounds on $H(s)$ often grow far too rapidly with $q$.

$\Longrightarrow$ SEVERELY impede uniformity in $q$.



# One of the main results...

**Theorem 1 (S.R. '25).** Fix $K_0 > 0$. Assume for $s$ with $\mathrm{Re}(s) > 1$, that $\sum_{n\geq 1} a_n/n^s = \left(\prod_{\chi \bmod q} L(s,\chi)^{\alpha_\chi}\right)\cdot G(s)$, where $\{\alpha_\chi\}_\chi \subset \mathbb{C}$, and $G(s)$ is well-behaved. Then for some $\{c_j\}_{j\geq 1} \subset \mathbb{C}$, we have **uniformly** in $x \geq 3$, $N \geq 0$ and $q \leq (\log x)^{K_0}$,

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- $q \leq (\log x)^{K_0}$ unconditional range in several applications **("Siegel–Walfisz range")**.
- Conditionally on Generalized Riemann Hypothesis/Landau–Siegel zeros conjecture $\implies$ Wider uniformity ranges in $q$.

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**Problem:** Given $q \in \mathbb{Z}^+$ and $\mathcal{A} \subset (\mathbb{Z}/q\mathbb{Z})^\times$, estimate

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✓ NOTE (for all applications): Better ranges on $q$ conditionally.

# Appl. 2. The Least Invariant Factor of Unit Groups

$(\mathbb{Z}/n\mathbb{Z})^\times \cong \mathbb{Z}/\lambda_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\lambda_r\mathbb{Z}$ with $\lambda_i \in \mathbb{Z}^+$ s.t. $\lambda_1 \mid \ldots \mid \lambda_r$.

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- **Theorem 1** $\Longrightarrow$ ✓Uniformity in $q \leq (\log x)^{K_0}$
  ✓Sharper asymptotics (improved error terms).

# Appl. 2. The Least Invariant Factor of Unit Groups

$(\mathbb{Z}/n\mathbb{Z})^\times \cong \mathbb{Z}/\lambda_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\lambda_r\mathbb{Z}$ with $\lambda_i \in \mathbb{Z}^+$ s.t. $\lambda_1 \mid \ldots \mid \lambda_r$.

**Least invariant factor:** $\boldsymbol{\lambda^*(n) := \lambda_1}$.

**Distribution of $\{\lambda^*(n)\}_{n\geq 1}$?** Estimate $\#\{n \leq x : \lambda^*(n) = q\}$.

- **Chang–Martin (2020):** Fixed $q \in \mathbb{Z}^+$.
- **Theorem 1** $\Longrightarrow$ ✓Uniformity in $q \leq (\log x)^{K_0}$
  ✓Sharper asymptotics (improved error terms).

**Theorem 1** $\Longrightarrow$ distribution of **least elementary divisor** of $(\mathbb{Z}/n\mathbb{Z})^\times$
$\rightsquigarrow$ extending work of **Martin–Nguyen (2024)**.

## Appl. 3. A sieving problem for multiplicative functions

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- **Theorem 1** $\Longrightarrow$ Extend to $q \le (\log x)^{K_0}$ and more general $f$.

# Appl. 4. Sathe-Selberg in arithmetic progressions

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$$\omega_a(n) := \sum_{p|n\colon p \equiv a \pmod{q}} 1, \qquad \Omega_a(n) := \sum_{p|n\colon p \equiv a \pmod{q}} v_p(n) \quad ?$$

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**Theorem 1** $\implies$ estimate $\#\{n \le x : f(n) = k\}$ for $f \in \{\omega_a, \Omega_a\}$, uniformly in $q \le (\log x)^{K_0}$ and in $a \in (\mathbb{Z}/q\mathbb{Z})^\times$.

## Appl. 5. Residue-class distribution of multiplicative functions

**Problem:** Given multiplicative $f : \mathbb{Z}^+ \to \mathbb{Z}$ and $q \in \mathbb{Z}^+$, $a \in (\mathbb{Z}/q\mathbb{Z})^\times$, estimate $\#\{n \le x : f(n) \equiv a \pmod{q}\}$.

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**General:** Given multiplicative $f_1, \ldots, f_K : \mathbb{Z}^+ \to \mathbb{Z}$ and $q \in \mathbb{Z}^+$, $a_i \in (\mathbb{Z}/q\mathbb{Z})^\times$, estimate $\#\{n \le x : (\forall i)\ f_i(n) \equiv a_i \pmod q\}$.

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**Aside (Ingredients):** "Mixing" in $(\mathbb{Z}/q\mathbb{Z})^\times$, detected via: anatomy of integers, character sum machinery, linear algebra over rings, arithmetic geometry, algebraic geometry, Theorem 1.

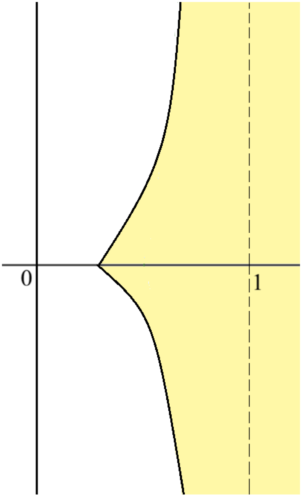
## Summary of the main ideas

**Set-up:** $\sum_{n\geq 1} a_n/n^s = \mathcal{F}(s)G(s)$, with

$$\mathcal{F}(s) := \prod_{\chi \bmod q} L(s,\chi)^{\alpha_\chi},$$

and $G(s)$ well-behaved.

**To estimate:** $$\sum_{n\leq x} a_n.$$



# Main ideas behind Theorem 1: Summarized

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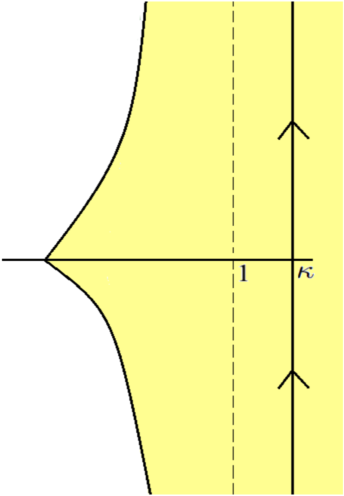
**Step 1. Perron's formula**

$$\sum_{n\leq x} a_n = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{\mathcal{F}(s)G(s)x^s}{s}\, ds.$$



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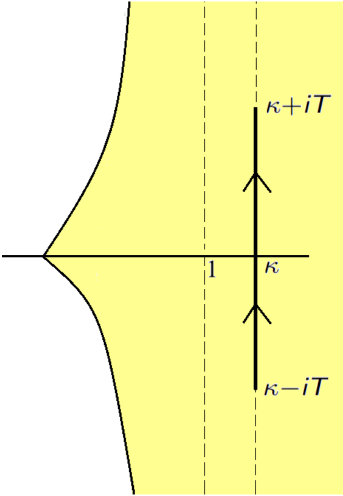

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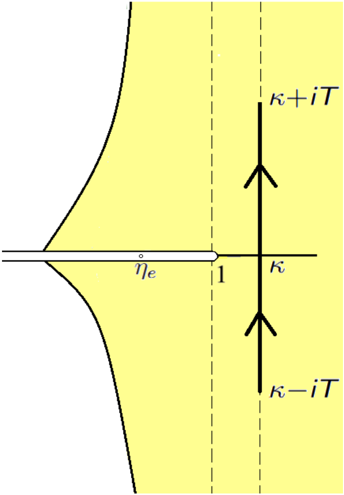
**Step 2.** $\mathcal{F}(s)$ analytically continues into the shaded region.

- Two possible branch points, viz. $s=1$ and $s=\eta_e$.



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By Cauchy's Integral Theorem,

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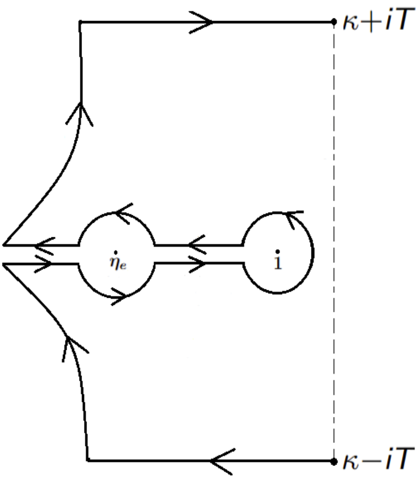

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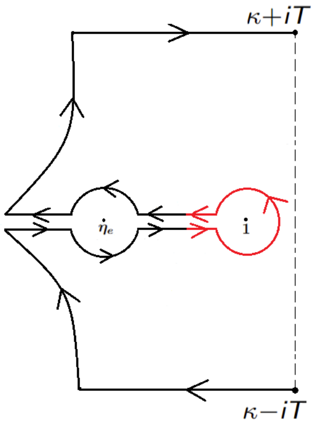


**Step 3.**

$$\int_{\text{Red part}} \frac{\mathcal{F}(s)G(s)x^s}{s}\, ds \to \text{main term, secondary term, } \ldots$$

## Bounding the contribution of the rest of contour Γ

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- Motivation: For $\mathrm{Re}(s) > 1$, we know that

$$\frac{L'(s,\chi)}{L(s,\chi)} = -\sum_n \frac{\chi(n)\Lambda(n)}{n^s}, \quad \text{where} \quad \Lambda(n) := \begin{cases} \log p, & \text{if } n = p^k \\ 0, & \text{otherwise.} \end{cases}$$

Hence with $\varrho(n) := \sum_{\chi \bmod q} \alpha_\chi \cdot \chi(n)$, we have for $\mathrm{Re}(s) > 1$,

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**Attempt:** Relate $\mathcal{F}'(s)/\mathcal{F}(s)$ with **finite truncations** of its Dirichlet series?

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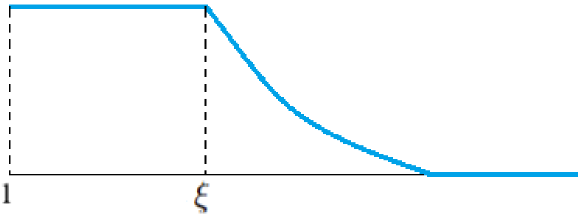
Relate $\mathcal{F}'(s)/\mathcal{F}(s)$ with a **continuous finite truncation** of its Dir. ser.?

- Want something like $\sum_n \varrho(n)\Lambda(n)w(n)/n^s$, where $w(x)$ looks like



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**First try:**

$$\text{For any } B > 2, \quad \frac{1}{2\pi i}\int_{B-i\infty}^{B+i\infty} \frac{X^z}{z}\,\mathrm{d}z = \begin{cases} 1, & \text{if } X > 1 \\ 1/2, & \text{if } X = 1 \\ 0, & \text{if } 0 < X < 1 \end{cases}$$

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$$\implies \text{For any } B > 2 + \mathrm{Re}(s), \ \frac{1}{2\pi i}\int_{B-i\infty}^{B+i\infty} \frac{X^{z-s}}{z-s}\,\mathrm{d}z = \begin{cases} 1, & \text{if } X > 1 \\ 1/2, & \text{if } X = 1 \\ 0, & \text{if } 0 < X < 1 \end{cases}$$

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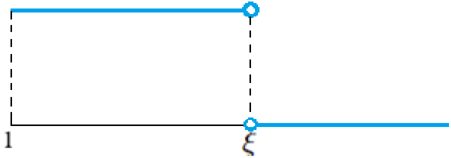
$$\frac{1}{2\pi i}\int_{B-i\infty}^{B+i\infty} \frac{\xi^{z-s}}{z-s}\cdot\frac{\mathcal{F}'(z)}{\mathcal{F}(z)}\,\mathrm{d}z \quad \text{“}=\text{”} \quad \sum_n \frac{\varrho(n)\,\Lambda(n)}{n^s}\cdot\frac{1}{2\pi i}\int_{B-i\infty}^{B+i\infty}\frac{(\xi/n)^{z-s}}{z-s}\,\mathrm{d}z$$

$$= \sum_{n<\xi}\frac{\varrho(n)\,\Lambda(n)}{n^s} + \frac{1}{2}\cdot\mathbb{1}_{\xi\in\mathbb{Z}^+}\frac{\varrho(\xi)\,\Lambda(\xi)}{\xi^s}.$$



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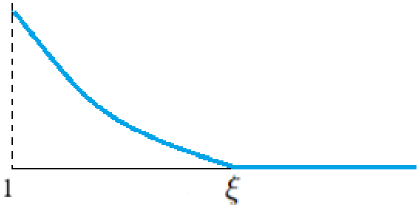

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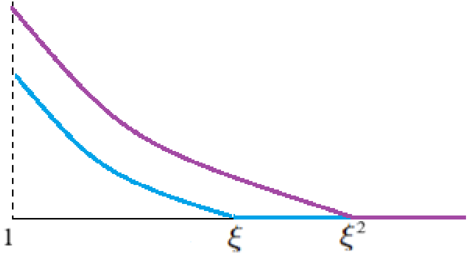

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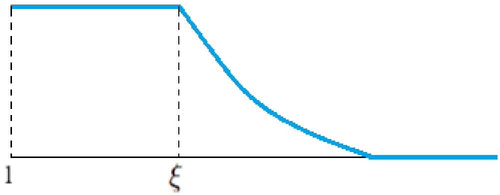

For $\xi := (q\,(|\mathrm{Im}\ s| + 1))^6$ and $B := 2 + |s|$, consider

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Another contour shift gives, for any $s \ne 1$ satisfying $L(s, \chi) \ne 0$,

$$\mathcal{I}(s) = \frac{\mathcal{F}'(s)}{\mathcal{F}(s)} - \frac{\alpha_{\chi_0}\,(\xi^{2(1-s)} - \xi^{1-s})}{(1-s)^2 \log \xi} + \sum_{\chi \bmod q} \sideset{}{^*}\sum_{\rho:\, L(\rho,\chi)=0} \frac{\alpha_\chi\,(\xi^{2(\rho-s)} - \xi^{\rho-s})}{(\rho-s)^2 \log \xi}$$

**Recall:** $\mathcal{F}'(s)/\mathcal{F}(s) = \sum_{\chi \bmod q} \alpha_\chi\, L'(s,\chi)/L(s,\chi)$ by defn of $\mathcal{F}$.

Hence for any $s \neq 1$ satisfying $L(s, \chi) \neq 0$, we have

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(i) Dirichlet polynomials:

Bounded via
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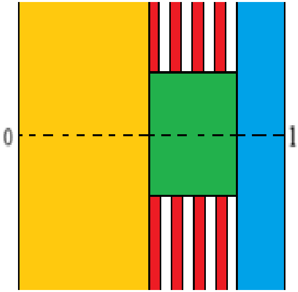
(ii) Double sum:

Split into regions + zero-density estimates.



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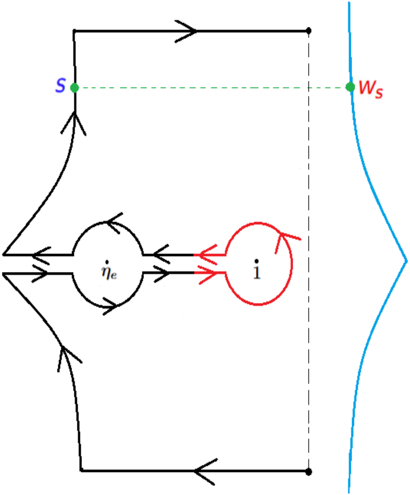
**Idea 2.**

Suitable bound on $\mathcal{F}'(s)/\mathcal{F}(s)$.

**Idea 3.** "Auxiliary functions".



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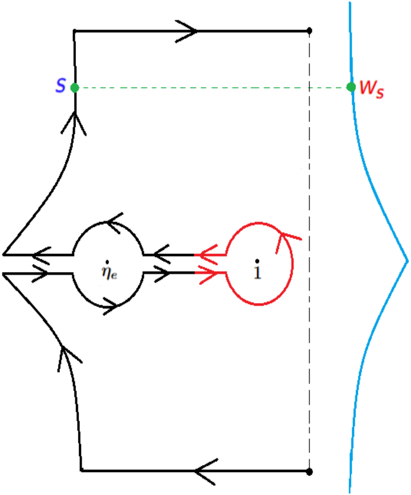

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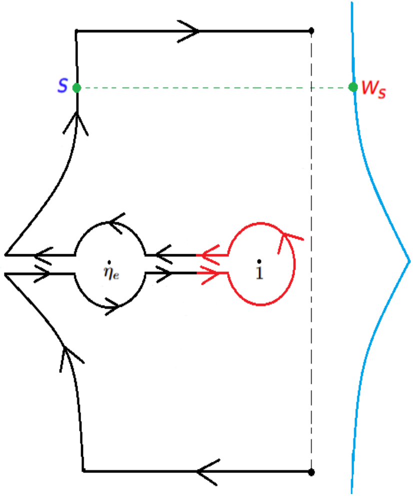

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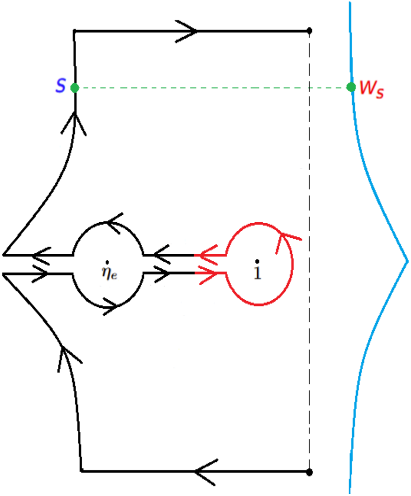

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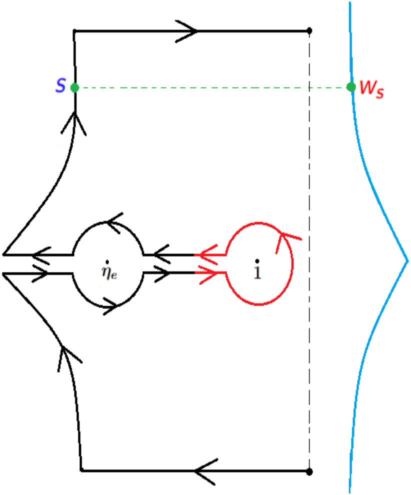

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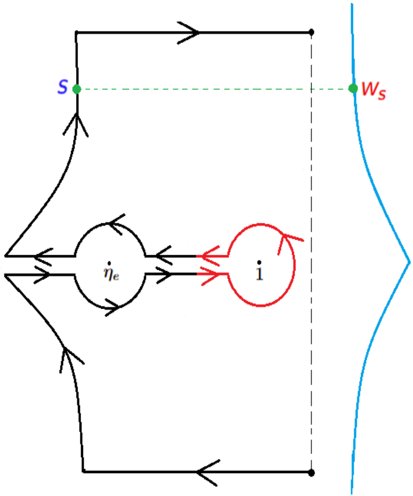
- Profit!
  Suitable bound on $\mathcal{F}(s)$ on rest of $\Gamma$.



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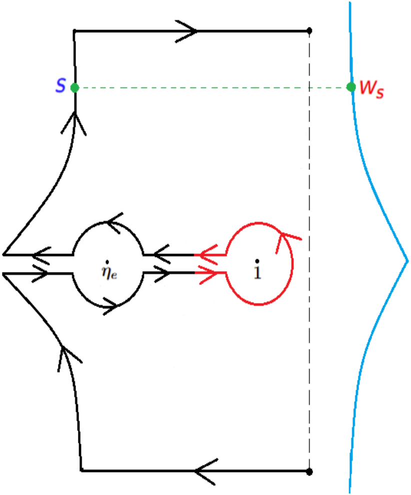
**Idea 3.**

Auxiliary functions $\implies$ Bound on $\mathcal{F}(s)$.

**General Case:** For some fixed $\nu > 0$ and for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/\nu$, assume

$$\sum_{n\geq 1} a_n/n^s = \left(\prod_{\chi \bmod q} L(s\nu,\chi)^{\alpha_\chi}\right) G(s)$$

$\rightsquigarrow$ scaling + averaging + "pigeonhole".



Thank you for your attention!

Email: akash01s.roy@gmail.com