THE LANDAU-SELBERG-DELANGE METHOD FOR PRODUCTS OF DIRICHLET L-FUNCTIONS

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1. Introduction

We write complex numbers s as $\sigma + it$, where $\sigma = \mathrm{Re}(s)$ and $t = \mathrm{Im}(s)$. Fix $\boldsymbol{c_0 \in (0, 1/3)}$ such that for any integer $q \geq 3$, the product $\prod_{\chi \bmod q} L(s, \chi)$ has no zero in the region $\{\sigma + it : \sigma > 1 - c_0/\log(q(|t|+1))\}$ except at most a simple real zero $\boldsymbol{\eta_e}$ (the "Siegel zero") associated to a real character $\boldsymbol{\chi_e}$ (the "exceptional character"). We also fix any $\boldsymbol{\nu > 0}$ and $\boldsymbol{\delta_0 \in (0, 1]}$, and define $\boldsymbol{\mathcal{L}_q(t) = \log(q(|t\nu| + 1))}$, $\boldsymbol{\mathcal{D}(c_0) = \{\sigma + it : \sigma > 1 - c_0/\mathcal{L}_q(t)\}}$, and

$$\boldsymbol{\lambda_q \ := \ 1 \ + \ \max_{a \bmod q} \ \max \left\{ \left|\sum_{\chi} \ \alpha_\chi \chi(a)\right|, \ \left|\sum_{\chi} \ \beta_\chi \chi(a)\right| \right\}}.$$

Writing $\alpha_\chi = \sum_{\psi \bmod q} \alpha_\psi \cdot \mathbb{1}_{\psi=\chi} = \sum_{\psi \bmod q} \alpha_\psi \cdot \varphi(q)^{-1} \sum_{a \bmod q} \overline{\chi}(a)\psi(a)$ and interchanging sums, we obtain the following important bound

$$|\alpha_\chi| \leq \lambda_q \ \text{ and } |\beta_\chi| \leq \lambda_q \text{ for all characters } \chi \bmod q. \tag{1.1}$$

Let $\{a_n\}_{n=1}^\infty$ be a sequence of complex numbers, $\{\alpha_\chi\}_{\chi \bmod q}$ be a set of complex numbers (indexed at the Dirichlet characters χ mod q), and $\Omega : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a non–decreasing function. We say that $\{a_n\}_{n=1}^\infty$ has property $\mathcal{P}(\{\alpha_\chi\}_\chi, c_0, \Omega)$ if the Dirichlet series $\sum_{n=1}^\infty a_n/n^s$ is of the form $\mathcal{F}(s\nu)G(s)$ for all complex numbers s having $\sigma > 1/\nu$, where $\mathcal{F}(s\nu) \coloneqq \prod_{\chi \bmod q} L(s\nu, \chi)^{\alpha_\chi}$, and where $G(s)$ is a function that analytically continues into the region $\mathcal{D}(c_0)$ and satisfies $|G(s)| \leq \Omega(t)$ therein. We shall also say that a positive integer N is good (with respect to $\{a_n\}_{n=1}^\infty$) if for any constant $c > 0$, there exists a constant $\kappa_c(N) > 0$ depending only on c and N such that

$$\sum_{x \leq n \leq x + cx/(\log x)^N} |a_n| \ \leq \ \kappa(c, N) \cdot \frac{x^{1/\nu}}{(\log x)^N} \quad \text{for all } x \geq 2.$$

Our first main result is the following:

Theorem 1.1. *Let $\{a_n\}_{n=1}^\infty$ be a sequence of complex numbers. We say that such that the Dirichlet series $\sum_{n=1}^\infty a_n/n^s$ has property $\mathcal{P}(\{\alpha_\chi\}_\chi, c_0, \Omega)$. Then uniformly in $x \geq 4$, in good $N \geq 0$, and in moduli $q \geq 4$ satisfying $(1 - \eta_e)\log x > 3\nu$, we have*

$$\sum_{n \leq x} a_n - \frac{x^{1/\nu}}{(\log x)^{1-\alpha_{\chi_0}}} \sum_{0 \leq j \leq N} \frac{1}{\Gamma(\alpha_{\chi_0} - j)} \cdot \frac{\kappa_j}{(\log x)^j}$$
$$\ll (4\lambda_q \log q)^{\lambda_q + 2K} \cdot \kappa_c(N) x^{1/\nu} \left\{ \frac{\Omega_{\mathrm{gr}}(T)(\log T)^{1+\lambda_q}}{T} \ + \ \frac{\Omega_{\mathrm{gr}}(1/\nu)(1-\eta_e)^{-2K}\Gamma(N + 2 + |\alpha_{\chi_0}|)}{(2(1-\eta_e)\log x/141\nu)^{N+2-\mathrm{Re}(\alpha_{\chi_0})}} \right\}.$$

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2. Key analytic inputs: Logarithmic Derivatives and Auxiliary Functions

For any χ mod q, the function $\mathrm{Log}L(s\nu,\chi) \coloneqq \sum_{p,r\ge 1} \chi(p^r)/rp^{rs\nu}$ defines an analytic logarithm of $L(s\nu,\chi)$ on the region $\{s : \sigma > 1/\nu\}$. Hence, the function $\mathcal{F}(s\nu)$ is analytic on $\{s : \sigma > 1/\nu\}$, and

$$\mathcal{F}(s\nu) = \prod_{\chi} L(s\nu,\chi)^{\alpha_\chi} = \exp\left(\sum_{\chi} \alpha_\chi \mathrm{Log}L(s\nu,\chi)\right) = \exp\left(\sum_{p,r\ge 1}\frac{1}{rp^{rs\nu}}\sum_{\chi}\alpha_\chi \chi(p^r)\right) \quad \text{if } \sigma > 1/\nu. \tag{2.1}$$

We now make it clear how our functions can be analytically continued into regions of interest. In what follows, anything involving the Siegel zero η_e is to be ignored if η_e doesn't exist.

2.1. **Analytic Continuations.** Since the functions $L(s\nu,\chi_0)(s-1/\nu)$, $L(s\nu,\chi_e)(s-\eta_e/\nu)^{-1}$, and $\{L(s\nu,\chi)\}_{\chi\ne\chi_0,\chi_e \bmod q}$ all continue analytically into nonvanishing functions on $\mathcal{D}(c_0)$, they have (unique) analytic logarithms $\mathcal{T}^*(s,\chi_0)$, $\mathcal{T}^*(s,\chi_e)$, and $\{\mathcal{T}(s,\chi)\}_{\chi\ne\chi_0,\chi_e \bmod q}$ on $\mathcal{D}(c_0)$ satisfying

$$\mathcal{T}^*\left(\frac{2}{\nu},\chi_0\right) = \sum_{p,r\ge 1}\frac{\chi_0(p^r)}{rp^{2r}} + \ln\left(\frac{2}{\nu}-\frac{1}{\nu}\right),\quad \mathcal{T}^*\left(\frac{2}{\nu},\chi_e\right) = \sum_{p,r\ge 1}\frac{\chi_e(p^r)}{rp^{2r}} - \ln\left(\frac{2}{\nu}-\frac{\eta_e}{\nu}\right),$$

and $\mathcal{T}(2/\nu,\chi) = \sum_{p,r\ge 1}\chi(p^r)/rp^{2r}$ for all other χ. (Thus $\mathcal{T}^*(s,\chi_0)$ is analytic on $\mathcal{D}(c_0)$ and satisfies $e^{\mathcal{T}^*(s,\chi_0)} = L(s\nu,\chi_0)(s-1/\nu)$ therein, etc.) Comparing derivatives, we see that the functions

$$\mathcal{T}(s,\chi_0) \coloneqq \mathcal{T}^*(s,\chi_0) - \log\left(s-\frac{1}{\nu}\right) \quad\text{and}\quad \mathcal{T}(s,\chi_e) \coloneqq \mathcal{T}^*(s,\chi_e) + \log\left(s-\frac{\eta_e}{\nu}\right) \tag{2.2}$$

define unique analytic continuations of the functions $\mathrm{Log}L(s,\chi_0)$ and $\mathrm{Log}L(s,\chi_e)$, into the regions $\mathcal{D}(c_0)\setminus(-\infty,1/\nu]$ and $\mathcal{D}(c_0)\setminus(-\infty,\eta_e/\nu]$, respectively. (Here $\log z$ is the principal branch of the logarithm, so $\log(s-1/\nu)$ is analytic on $\mathbb{C}\setminus(-\infty,1/\nu]$.) From this discussion, we see that the function $\exp(\sum_\chi \alpha_\chi\mathcal{T}(s,\chi)) = \prod_\chi e^{\alpha_\chi\mathcal{T}(s,\chi)}$ defines a unique analytic continuation of $\mathcal{F}(s\nu)$ in (2.1) into $\mathcal{D}(c_0)\setminus(-\infty,1/\nu]$; hence, $\mathcal{F}(s\nu) = \exp(\sum_\chi \alpha_\chi\mathcal{T}(s,\chi))$ for all s in this region.

Note also that by the first equality in (2.1) and by analytic continuation, we may write

$$\frac{\mathcal{F}'(s\nu)}{\mathcal{F}(s\nu)} = \sum_{\chi}\alpha_\chi\frac{L'(s\nu,\chi)}{L(s\nu,\chi)} \text{ for all } s\ne 1/\nu \text{ s.t. } s\ne\rho/\nu \text{ for any complex zero } \rho \text{ of } \prod_\chi L(s,\chi). \tag{2.3}$$

This relation is consistent with the analytic continuation of $\mathcal{F}(s\nu)$ in the previous paragraph.

We will also need the following two auxiliary functions: By the above discussion (especially (2.2)),

- The function $\widetilde{\mathcal{H}}(s) \coloneqq \exp\left(\alpha_{\chi_0}\mathcal{T}^*(s,\chi_0) + \alpha_{\chi_e}\mathcal{T}^*(s,\chi_e) + \sum_{\chi\ne\chi_0,\chi_e}\alpha_\chi\mathcal{T}(s,\chi)\right)$ analytically continue the function $\mathcal{F}(s\nu)(s-1/\nu)^{\alpha_{\chi_0}}(s-\eta_e/\nu)^{-\alpha_{\chi_e}}$ into the region $\mathcal{D}(c_0)$.
- The function $\mathcal{H}(s) \coloneqq s^{-1}\exp\left(\alpha_{\chi_0}\mathcal{T}^*(s,\chi_0) + \sum_{\chi\ne\chi_0}\alpha_\chi\mathcal{T}(s,\chi)\right)$ analytically continues the function $s^{-1}\mathcal{F}(s\nu)(s-1/\nu)^{\alpha_{\chi_0}}$ into the region $\mathcal{D}(c_0)\setminus(-\infty,\eta_e/\nu]$.

The reader may now forget all the $\mathcal{T}$ and $\mathcal{T}^*$. All that needs to be remembered from this subsection are (2.1) and (2.3), that $\mathcal{F}(s\nu)$ continues analytically into $\mathcal{D}(c_0)\setminus(-\infty,1/\nu]$, and that

$$\widetilde{\mathcal{H}}(s) = \mathcal{F}(s\nu)(s-1/\nu)^{\alpha_{\chi_0}}(s-\eta_e/\nu)^{-\alpha_{\chi_e}} \qquad \text{for all } s\in\mathcal{D}(c_0), \tag{2.4}$$

$$\mathcal{H}(s) = s^{-1}\mathcal{F}(s\nu)(s-1/\nu)^{\alpha_{\chi_0}} \qquad \text{for all } s\in\mathcal{D}(c_0)\setminus(-\infty,\eta_e/\nu], \tag{2.5}$$

with $\widetilde{\mathcal{H}}(s)$ and $\mathcal{H}(s)$ being analytic on $\mathcal{D}(c_0)$ and $\mathcal{D}(c_0)\setminus(-\infty,\eta_e/\nu]$, respectively.

2.2. **Analysis of Logarithmic Derivatives.** To give suitable bounds on $\mathcal{F}(s\nu)$, we will first analyze its logarithmic derivative. To this end, the following known results on Dirichlet L-functions will be useful. In what follows, we write $\boldsymbol{\rho = \beta + i\gamma}$ where $\boldsymbol{\beta = \mathrm{Re}(\rho)}$ and $\boldsymbol{\gamma = \mathrm{Im}(\rho)}$. We denote by $\sum^{*}_{\rho\colon L(\rho,\chi)=0}$ a sum over all zeros ρ of $L(s,\chi)$ counted with appropriate multiplicity.

Lemma 2.1. *The following hold uniformly in $q \geq 2$ and in* ***all*** *Dirichlet characters χ mod q.*

(1) *Uniformly in all real t, we have* $\displaystyle\sum^{*}_{\substack{\rho\colon L(\rho,\chi)=0\\ 0\leq\beta\leq1}} \frac{1}{1+(t-\gamma)^2} \ll \log(q(|t|+1))$.

(2) *Uniformly in all complex s satisfying $\sigma \in [-1,2]$, $|t| \geq 2$, and $t \neq \gamma$ for any of the zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$, we have* $\displaystyle\frac{L'(s,\chi)}{L(s,\chi)} = \sum^{*}_{\substack{\rho\colon\ L(\rho,\chi)=0\\ 0\leq\beta\leq1,\ |\gamma-t|\leq1}} \frac{1}{s-\rho} + O(\log(q(|t|+1)))$.

(3) *We have $L'(s,\chi)/L(s,\chi) \ll \log(q|s|)$, uniformly in all complex s satisfying $\sigma \leq -1$ and lying outside the disks of radius $1/4$ about the trivial zeros of $L(s,\chi)$.*

(4) *Uniformly in real $t \notin (-1,1)$, we have $\#\{\rho : 0 \leq \beta \leq 1,\ |\gamma - t| \leq 1,\ L(\rho,\chi) = 0\} \ll \log(q|t|)$.*

In most standard texts, these results are stated and proved only for primitive characters, however the generality above will be helpful here. (Section 8 discusses this lemma for general χ mod q.)

We now give a certain (absolutely convergent) series expansion for the logarithmic derivative of $\mathcal{F}(s\nu)$ in terms of the zeros of the L-functions, with coefficients that are easy to control.

Proposition 2.2. *For any $s \in \mathbb{C}$ satisfying $s \neq 1/\nu$ and $s \neq \rho/\nu$ for any zero ρ of $\prod_\chi L(s,\chi)$,*

(2.6)
$$\frac{\mathcal{F}'(s\nu)}{\mathcal{F}(s\nu)} = \sum_{n\leq\xi^2} \frac{\varrho(n)\Lambda(n)}{n^{s\nu}}\tau(n) - \frac{\alpha_{\chi_0}(\xi^{1-\nu s} - \xi^{2(1-\nu s)})}{(1-\nu s)^2\log\xi} + \sum_{\chi \bmod q}\ \sum^{*}_{\rho\colon L(\rho,\chi)=0} \frac{\alpha_\chi(\xi^{\rho-\nu s} - \xi^{2(\rho-\nu s)})}{(\rho-\nu s)^2\log\xi},$$

where $\xi := e^{6\mathcal{L}_q(t)}$, $\varrho(n) := \sum_{\chi \bmod q} \alpha_\chi\, \chi(n)$, and $\tau(n) := \mathbb{1}_{n\leq\xi} + \mathbb{1}_{\xi<n\leq\xi^2}\,(2 - \log n/\log\xi)$.

Proof. Our starting point is the identity $\int_{b-i\infty}^{b+i\infty} y^z/z^2\,\mathrm{d}z = \mathbb{1}_{y>1}\cdot 2\pi i\log y$ which holds for any $b, y > 0$. To see this, consider any $R \geq 2$, apply the residue theorem to the contour consisting of the vertical segment $[b - iR, b + iR]$ and the major arc (respectively, minor arc) of the circle centered at the origin passing through $b \pm iR$ if $y > 1$ (resp. $y \leq 1$), and then let $R \to \infty$.

The Dirichlet series of $L'(s,\chi)/L(s,\chi)$ and (2.3) give $\mathcal{F}'(z\nu)/\mathcal{F}(z\nu) = \sum_{n\geq1}\varrho(n)\Lambda(n)/n^{z\nu}$ for all z with $\mathrm{Re}(z) > 1/\nu$. We now claim that for all s as in the statement of the proposition,

$$\text{(2.7)}\qquad \frac{1}{2\pi i}\int_{\frac{2}{\nu}+|s|-i\infty}^{\frac{2}{\nu}+|s|+i\infty} \frac{\xi^{\nu(z-s)} - \xi^{2\nu(z-s)}}{(z-s)^2}\cdot\frac{\mathcal{F}'(z\nu)}{\mathcal{F}(z\nu)}\,\mathrm{d}z = \nu\sum_{n\leq\xi^2}\frac{\varrho(n)\Lambda(n)}{n^{z\nu}}\tau(n)\log\xi.$$

Indeed by the identity in the first paragraph of the proof, (2.7) is immediate if $\mathcal{F}'(z\nu)/\mathcal{F}(z\nu)$ were replaced by any finite truncation $\sum_{n\leq Y}\varrho(n)\Lambda(n)/n^{z\nu}$ of its aforementioned Dirichlet series (for any $Y > \xi^6$). Moreover by the same Dirichlet series, the absolute value of the integrand above is at most $2\lambda_q\,\xi^{4+2\nu|s|}\,(\sum_n \Lambda(n)/n^2)\,|z-s|^{-2}$, which is an L^1-function since $\int_{2/\nu+|s|-i\infty}^{2/\nu+|s|+i\infty}|\mathrm{d}z|/|z-s|^2 < \infty$ and $\sum_n \Lambda(n)/n^2 \ll 1$. Thus (2.7) follows by the Dominated Convergence Theorem.

We will now shift contours. For this, note that for any $M \geq 2$, the number of zeros of $\prod_\chi L(s,\chi)$ in the rectangle $[0,1] \times (M, M+1]$ is $\ll \varphi(q)\log(qM)$ by Lemma 2.1(4). Hence there exists $T_M \in (M, M+1]$ satisfying $|T_M - \gamma| \gg (\varphi(q)\log(qM))^{-1}$ for **all** zeros $\rho = \beta + i\gamma$ of $\prod_\chi L(s,\chi)$. Since the set of zeros of $\prod_\chi L(s,\chi)$ is closed under complex conjugation, this also means that

$$|T_M \pm \gamma| \gg (\varphi(q)\log(qM))^{-1} \text{ for all zeros } \rho = \beta + i\gamma \text{ of } \prod_\chi L(s,\chi). \tag{2.8}$$

With the contour ω_M as in Figure 1, we claim that

$$\frac{L'(z\nu,\chi)}{L(z\nu,\chi)} \ll \varphi(q)\log^2(qM), \text{ uniformly in } q \geq 3,\ \chi \bmod q,\ M \geq 2(1+\nu+\nu|s|),\ z \in \omega_M. \tag{2.9}$$

If $\mathrm{Re}(z) \geq 2/\nu$, this follows from the Dirichlet series of $L'(z\nu,\chi)/L(z\nu,\chi)$. If $\mathrm{Re}(z) \in [-1/\nu, 2/\nu]$, then z must lie on the two horizontal segments in ω_M, so that by (2.8), we have $|z\nu - \rho| \geq |\mathrm{Im}(z)\nu - \gamma| = |T_M \pm \gamma| \gg (\varphi(q)\log(qM))^{-1}$ for any zero $\rho = \beta + i\gamma$ of $\prod_\chi L(s,\chi)$. This gives (2.9) by Lemma 2.1(2) and (4). Lastly if $\mathrm{Re}(z) \leq -1/\nu$, then Lemma 2.1(3) establishes (2.9).

Now for any $M \geq 2\nu|s|$ and any $z \in \omega_M$, we have $|z - s| \geq |z| - |s| \geq |z|/2 \geq M/2\nu$. As such $\int_{\omega_M} |\mathrm{d}z|/|z-s|^2 \ll_{\nu,s} \int_{M/2\nu}^\infty \mathrm{d}t/t^2 + (M/2\nu)^{-2} \cdot M \ll M^{-1}$, so that (2.3) and (2.9) yield

$$\lim_{M\to\infty} \int_{\omega_M} \frac{\xi^{\nu(z-s)} - \xi^{2\nu(z-s)}}{(z-s)^2} \cdot \frac{\mathcal{F}'(z\nu)}{\mathcal{F}(z\nu)}\,\mathrm{d}z \;=\; 0. \tag{2.10}$$

Using the residue theorem to shift contours from the vertical line in (2.7) to ω_M, and then letting $M \to \infty$, we thus find from (2.10), (2.7) and (2.3) that

(2.11)

$$\nu \sum_{n \leq \xi^2} \frac{\varrho(n)\Lambda(n)}{n^{z\nu}} \tau(n)\log\xi \;=\; \left(\operatorname*{Res}_{z=s} + \operatorname*{Res}_{z=1/\nu} + \sum_{\rho:\ \prod_\chi L(\rho,\chi)=0} \operatorname*{Res}_{z=\rho/\nu} \right) \frac{\xi^{\nu(z-s)} - \xi^{2\nu(z-s)}}{(z-s)^2} \cdot \frac{\mathcal{F}'(z\nu)}{\mathcal{F}(z\nu)}.$$

Finally, using (2.3) to compute the above residues, we obtain the proposition. For instance, note that if $\xi^{\rho-\nu s} \neq 1$ for some ρ above, then (2.3) shows that $z = \rho/\nu$ is a simple pole of the function on the right of (2.11) of residue $\nu(\xi^{\rho-\nu s} - \xi^{2(\nu-\rho s)})(\rho - \nu s)^{-2} \sum_\chi \alpha_\chi \cdot \{\text{multiplicity of } \rho \text{ in } L(s,\chi)\}$. If $\xi^{\rho-\nu s} = 1$, then $z = \rho/\nu$ is a removable singularity, so we can still give the same expression (whose value is zero) for its "residue". The residue at $z = 1/\nu$ can be computed analogously, and the residue at $z = s$ (which is always necessarily a simple pole) is equal to $-\nu(\log\xi)\mathcal{F}'(s\nu)/\mathcal{F}(s\nu)$. $\square$

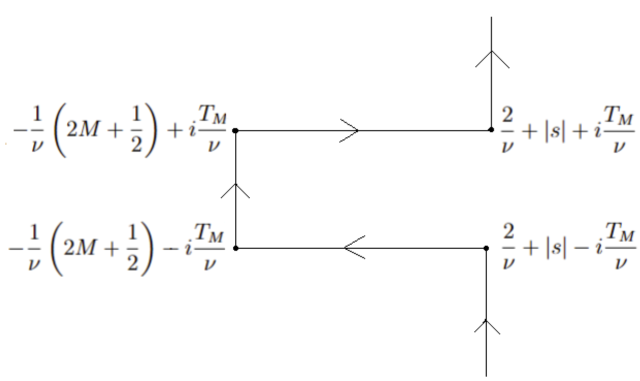

FIGURE 1. The Contour ω_M

We will now use the series representation in Proposition 2.2 to give a suitable bound on $\mathcal{F}'(s\nu)/\mathcal{F}(s\nu)$. A crucial input will be provided the following zero density estimate. In what follows, we define

$$\boldsymbol{N(\theta,t)} \;\; := \;\; \sum_{\chi \bmod q} \;\; \sideset{}{^*}\sum_{\substack{\rho:\; L(\rho,\chi)=0\\ \theta\le\beta\le 1,\; |\gamma|\le t}} \mathbf{1}.$$

Lemma 2.3. *We have* $N(\theta,t) \ll (qt)^{3(1-\theta)}$*, uniformly in* $q\ge 3$*,* $\theta\in[1/2,1]$*, and* $t\ge 1$*.*

This may be found in work of Heath–Brown and Jutila. We now state the bound alluded to above.

Proposition 2.4. *Uniformly in* $q\ge 3$*, and in complex numbers* s *satisfying* $\sigma \ge \nu^{-1}(1-c_0/2\mathcal{L}_q(t))$*,*

$$\left|\frac{\mathcal{F}'(s\nu)}{\mathcal{F}(s\nu)} \;+\; \frac{\alpha_{\chi_0}}{s\nu-1} \;-\; \frac{\alpha_{\chi_e}}{s\nu-\eta_e}\right| \;\ll\; \lambda_q\,\mathcal{L}_q(t).$$

Proof. Most of the argument consists of carefully bounding the different components of the right of (2.6). First, for all $n\le\xi^2$, we have $|n^{s\nu}| = n^{\sigma\nu} \ge n^{1-c_0/2\mathcal{L}_q(t)} \ge n\exp(-2\log\xi/2\mathcal{L}_q(t)) \gg n$, so that the first sum on the right in (2.6) is $\ll \lambda_q \sum_{n\le\xi^2} \Lambda(n)/n \ll \lambda_q\,\mathcal{L}_q(t)$ by Mertens' Theorem.

Next, since the trivial zeros of any $L(s,\chi)$ are simple, the total contribution of all zeros $\{-r/\nu\}_{r\in\mathbb{N}}$ to the right of (2.6) equals $(\log\xi)^{-1}\sum_{r\ge1}\left(\sum_{\chi:\;\chi(-1)=(-1)^r}\alpha_\chi\right)(\xi^{-(r+\nu s)}-\xi^{-2(r+\nu s)})(r+\nu s)^{-2}$. Since the sum on χ is $\sum_\chi \alpha_\chi(1+\chi(-1)(-1)^r)/2 = (\varrho(1)+\varrho(-1))/2$, it follows that the last expression has size $\ll \lambda_q(\log\xi)^{-1}\sum_{r\ge1}\xi^{-(r+\nu\sigma)}(r+\nu\sigma)^{-2} \ll \lambda_q\,\mathcal{L}_q(t)^{-1}\sum_{r\ge1} r^{-2} \ll \lambda_q\,\mathcal{L}_q(t)^{-1}$.

Now, we observe that $|(\xi^{\theta-\nu s}-\xi^{2(\theta-\nu s)})(\theta-\nu s)^{-2}(\log\xi)^{-1} \;-\; (\nu s-\theta)^{-1}| \ll \mathcal{L}_q(t)$ uniformly in $\theta\in(0,1]$ and s as in the proposition. This follows by a straightforward crude bounding if $|\theta-\nu s| > (\log\xi)^{-1}$, and by the formula $\xi^{\theta-\nu s} = 1-(\theta-\nu s)\log\xi + O\big((\theta-\nu s)^2(\log\xi)^2\big)$ if $|\theta-\nu s|\le(\log\xi)^{-1}$. Collecting all the observations made so far, we see that this proposition would follow from (2.6), once we show that uniformly in all s with $\sigma\ge\nu^{-1}(1-c_0/2\mathcal{L}_q(t))$,

$$\frac{1}{\mathcal{L}_q(t)}\sum_\chi |\alpha_\chi| \sideset{}{^*}\sum_{\substack{\rho:\; L(\rho,\chi)=0\\ 0\le\beta\le1,\;\rho\ne\eta_e}} \frac{\xi^{\beta-\nu\sigma}+\xi^{2(\beta-\nu\sigma)}}{(\beta-\nu\sigma)^2+(\gamma-\nu t)^2} \;\ll\; \lambda_q\mathcal{L}_q(t). \tag{2.12}$$

To show this, we start by bounding the entire expression above by $S_1+S_2+S_3+S_4$, where

- S_1 denotes the total contribution of all ρ having $\beta\le 1/2$, so that
$$S_1 = \frac{1}{\mathcal{L}_q(t)}\sum_\chi |\alpha_\chi| \sideset{}{^*}\sum_{\substack{\rho:\; L(\rho,\chi)=0\\ 0\le\beta\le1/2}} \frac{\xi^{\beta-\nu\sigma}+\xi^{2(\beta-\nu\sigma)}}{(\beta-\nu\sigma)^2+(\gamma-\nu t)^2}.$$
- S_2 denotes the total contribution of all ρ having $\beta\in(1/2,1]$ and $|\gamma|\le 2|t\nu|+1$.
- S_3 denotes the total contribution of all ρ having $\beta\in(1/2,\sigma\nu]$ and $|\gamma|> 2|t\nu|+1$.
- S_4 denotes the total contribution of all ρ having $\beta\in(\sigma\nu,1]$.

For any ρ appearing in S_1, we have $\beta-\nu\sigma\le 1/2-(1-c_0/2\mathcal{L}_q(t)) \le -1/2+1/2\log q\le -1/3$, so that $(\beta-\nu\sigma)^2+(\gamma-\nu t)^2 \ge (1+(\gamma-\nu t)^2)/9$. Hence (1.1) and Lemma 2.1(1) yield $S_1 \ll \lambda_q\,\xi^{1/2-\nu\sigma}\mathcal{L}_q(t)^{-1}\sum_\chi\sum_\rho(1+(\gamma-\nu t)^2)^{-1} \ll \lambda_q\cdot q\xi^{1/2-\nu\sigma} \ll \lambda_q\cdot q\xi^{-1/2}\cdot\xi^{c_0/2\mathcal{L}_q(t)} \ll \lambda_q$.

For any ρ appearing in S_3, we have $\beta - \nu\sigma \le 0$ and $|\gamma - t\nu| \ge |\gamma| - |t\nu| \ge |\gamma|/2$. Thus by (1.1),

$$S_3 \le \frac{8\lambda_q}{\mathcal{L}_q(t)} \sum_{\chi} \sideset{}{^*}\sum_{\substack{\rho \ne \eta_e:\ L(\rho,\chi)=0 \\ |\gamma|>2|t\nu|+1,\ 1/2<\beta\le\min\{\sigma\nu,1\}}} \xi^{\beta-\nu\sigma} \cdot |\gamma|^{-2}.$$

Partitioning the interval $(1/2, \min\{\sigma\nu, 1\}]$ into $R \coloneqq \lfloor \log \xi/2 \rfloor$ equally spaced intervals, we obtain

$$S_3 \le \frac{8\lambda_q}{\mathcal{L}_q(t)} \sum_{r=1}^{R} \xi^{1/2+r\mu_0/R-\nu\sigma} \sum_{\chi} \sideset{}{^*}\sum_{\substack{\rho\ne\eta_e:\ L(\rho,\chi)=0,\ |\gamma|>2|t\nu|+1 \\ 1/2+(r-1)\mu_0/R<\beta\le 1/2+r\mu_0/R}} |\gamma|^{-2}, \tag{2.13}$$

where $\mu_0 \coloneqq \min\{\sigma\nu, 1\} - 1/2$. Now the inner double sum (on χ and ρ) above is at most

$$\int_{2|t\nu|+1}^{\infty} \frac{\mathrm{d}N\left(\frac{1}{2} + \frac{(r-1)\mu_0}{R}, u\right)}{u^2} \ll \int_{2|t\nu|+1}^{\infty} \frac{N\left(\frac{1}{2} + \frac{(r-1)\mu_0}{R}, u\right)}{u^3}\,\mathrm{d}u \ll q^{3(1/2-(r-1)\mu_0/R)} \tag{2.14}$$

where we have used the Stieltjes integration by parts and Lemma 2.3. The last expression above is $\ll \xi^{1/4-(r-1)\mu_0/2R} \ll \xi^{1/4-r\mu_0/2R}$, as $\mu_0 \le 1/2$, $\xi \ge q^6$ and $R \ge \log\xi/3$. Inserting these into (2.13), we get $S_3 \ll \lambda_q\,\mathcal{L}_q(t)^{-1}\,\xi^{3/4-\nu\sigma}\ \sum_{r=1}^{R}\ \xi^{r\mu_0/2R}\ \le\ \lambda_q\,\mathcal{L}_q(t)^{-1}\ \xi^{3/4-\nu\sigma}\cdot R\xi^{\mu_0/2} \ll \lambda_q\,\xi^{1-\nu\sigma} \ll \lambda_q$.

Next, for any ρ counted in S_2, we have $|\gamma| \le 2|t\nu|+1$ and $\rho \ne \eta_e$, so that $\beta \le 1-c_0/\log(q(|\gamma|+1)) \le 1 - c_0/\log(2q(|t\nu|+1))$. Since $\nu\sigma \ge 1 - c_0/2\mathcal{L}_q(t)$, we get

$$\nu\sigma - \beta \ge c_0\left(\frac{1}{\log(2q(|t\nu|+1))} - \frac{1}{2\log(q(|t\nu|+1))}\right) = \frac{c_0}{\mathcal{L}_q(t)}\left(1 - \frac{\log 4}{\log(2q(|t\nu|+1))}\right) \ge \frac{c_0}{10\mathcal{L}_q(t)}.$$

Hence $(\beta-\nu\sigma)^2 \gg \mathcal{L}_q(t)^{-2}$. Proceeding as in (2.14) (via Lemma 2.3 and integration by parts),

$$\begin{aligned} S_2 &\ll \lambda_q\,\mathcal{L}_q(t) \sum_{\chi} \sideset{}{^*}\sum_{\substack{\rho\ne\eta_e:\ L(\rho,\chi)=0 \\ 1/2<\beta\le 1,\ |\gamma|\le 2|t\nu|+1}} \xi^{\beta-\nu\sigma}\ \le\ \lambda_q\,\mathcal{L}_q(t)\left(-\int_{1/2}^{1} \xi^{\theta-\nu\sigma}\,\mathrm{d}N(\theta, 2|t\nu|+1)\right) \\ &\le\ \lambda_q\,\mathcal{L}_q(t)\left(\xi^{1/2-\nu\sigma}\,N(1/2, 2|t\nu|+1)\ +\ \log\xi \int_{1/2}^{1} \xi^{\theta-\nu\sigma}\,N(\theta, 2|t\nu|+1)\,\mathrm{d}\theta\right) \\ &\ll\ \lambda_q\,\mathcal{L}_q(t)\left(\xi^{3/4-\nu\sigma}\ +\ \log\xi \int_{1/2}^{1} \xi^{(1+\theta)/2-\nu\sigma}\,\mathrm{d}\theta\right) \ll\ \lambda_q\,\mathcal{L}_q(t)\,\xi^{1-\nu\sigma}\ \ll \lambda_q\,\mathcal{L}_q(t). \end{aligned}$$

For any ρ ($\ne \eta_e$) in S_4, we have $1 - c_0/\log(q(|\gamma|+1)) \ge \beta > \sigma\nu \ge 1 - c_0/2\mathcal{L}_q(t)$, giving $|\gamma| > q(|t\nu|+1)^2 - 1$. Thus also $|\gamma - t\nu| \ge |\gamma| - |t\nu| \ge |\gamma|/2$. Proceeding exactly as we did for S_3,

$$S_4 \ll \frac{\lambda_q\,\xi^{2(1-\nu\sigma)}}{\mathcal{L}_q(t)} \sum_{\chi} \sideset{}{^*}\sum_{\substack{\rho:\,L(\rho,\chi)=0 \\ \sigma\nu<\beta<1,\ |\gamma|>q(|t\nu|+s1)^2-1}} |\gamma|^{-2} \ll \frac{\lambda_q}{\mathcal{L}_q(t)} \int_{q(|t\nu|+1)^2-1}^{\infty} \frac{\mathrm{d}N(\sigma\nu, u)}{u^2} \ll \lambda_q.$$

Collecting all these estimates establishes (2.12), completing the proof of the proposition. $\square$

The following is an important consequence of (2.4) and Proposition 2.4.

Corollary 2.5. *We have $\widetilde{\mathcal{H}}'(s)/\widetilde{\mathcal{H}}(s) \ll \lambda_q\,\mathcal{L}_q(t)$ uniformly in complex s having $\sigma \ge 1-c_0/2\mathcal{L}_q(t)$.*

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