Joint distribution in residue classes of families of multiplicative functions

Akash Singha Roy, University of Georgia
Partly based on joint work with
Paul Pollack and Noah Lebowitz-Lockard

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Example (Pillai, Selberg): $\Omega(n) = \sum_{p^k \| n} k$ is equidistributed mod q for each fixed q.

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Fact (Landau): For a fixed q, $\varphi(n) \equiv 0 \pmod q$ for "almost all" $n \in \mathbb{Z}^+$:

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For multiplicative functions $f : \mathbb{Z}^+ \to \mathbb{Z}$, it makes sense to study their distribution in the multiplicative group U_q mod q. So now our sample space is $\{n : \gcd(f(n), q) = 1\}$.

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- Consequence of a general criterion for weak equidistribution of a single "polynomially-defined" multiplicative function to a **fixed** modulus.

Explicit numerical distributions of $\varphi(n)$ mod 5:

For $x \geq 1$ and $r \in \{1,2,3,4\}$ let

$$\rho_r(x) := \frac{\#\{n \leq x : \varphi(n) \equiv r \pmod 5\}}{\#\{n \leq x : \gcd(\varphi(n),5) = 1\}}$$

x	$\rho_1(x)$	$\rho_2(x)$	$\rho_3(x)$	$\rho_4(x)$
10^5	0.27165	0.28003	0.23993	0.20837
10^6	0.27157	0.27556	0.23979	0.21307
10^7	0.27073	0.27267	0.23999	0.21660
10^8	0.26998	0.27051	0.24032	0.21917
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What fails mod 3? The numbers $p-1$, for $p \neq 3$ prime, either fail to be coprime to 3 or are "trapped" in the trivial subgroup of $(\mathbb{Z}/3\mathbb{Z})^\times$.

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Theorem 2 (Dence–Pomerance, 1998).

For $r \in \{-1,1\}$, we have as $x \to \infty$,

$$\#\{n \leq x : \varphi(n) \equiv r \pmod 3\} \sim c_r x/\sqrt{\log x},$$

where $c_1 \approx 0.6109$ and $c_{-1} \approx 0.3284$.

Analogously, $f_1, \cdots, f_K : \mathbb{Z}^+ \to \mathbb{Z}$ are **jointly weakly equidistributed** (or **jointly WUD**) modulo $q \in \mathbb{Z}^+$ if:

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Theorem 3.
$(\varphi, \sigma, \sigma_2)$ are jointly WUD modulo any fixed q s.t. $P^-(q) > 23$.

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Model (Siegel-Walfisz Theorem). Fix $K_0 > 0$. The primes $\leq x$ are weakly equidistributed mod q, uniformly for $q \leq (\log x)^{K_0}$. That is,

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Question (made precise). Can we establish analogues of Siegel-Walfisz with primes replaced by values of multiplicative functions or their families?

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Shortcomings:

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Shortcomings:

- Arguments restricted to a single multiplicative function and do not generalize to families, so could not uniformize Narkiewicz's 1982-criterion.
- Even for a single multiplicative function, we are not able to recover the full uniform version of Narkiewicz's 1967-criterion (for a single function) as we need to impose several additional restrictions on the modulus and on the multiplicative function.

Recent work (2023-'24): Removes all these limitations.

- Extended Narkiewicz's results to a varying modulus q optimally in almost every aspect (in particular optimal in the range and arithmetic restrictions on q).

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Consequences for $(\varphi, \sigma, \sigma_2)$

Theorem 5 (S.R., 2023-'24).

Fix $\epsilon \in (0,1)$. The family $(\varphi, \sigma, \sigma_2)$ is jointly WUD uniformly modulo $q \le (\log x)^{c_q}$ having $P^-(q) > 23$ and in $a_i \in U_q$, where $c_q > 0$ is a **small** *parameter depending on q.*

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Issue: $(\varphi, \sigma, \sigma_2)$ are **not** jointly WUD uniformly to all $q \le (\log x)^{K_0}$. Inputs n without many large prime factors obstruct uniformity!

Extending uniformity to the Siegel–Walfisz range:

Work-around: Restrict to inputs n having sufficiently many large prime factors. Equidistribution is restored among these inputs.

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Theorem 6 (S.R., 2023-'24).

Fix $K_0 > 0$ and $\epsilon \in (0,1)$. We have

$$\#\{n \le x : P_{13}(n) > q, (\varphi, \sigma, \sigma_2)(n) \equiv (a_1, a_2, a_3) \pmod q\} \\ \sim \frac{1}{\varphi(q)^3} \#\{n \le x : P_{13}(n) > q, \gcd(\varphi\sigma\sigma_2(n), q) = 1\},$$

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as $x \to \infty$, uniformly in $q \le (\log x)^{K_0}$ satisfying $P^-(q) > 23$ and in $a_i \in U_q$. For squarefree q, "13" can be replaced by "7".

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- Detect this "mixing" using methods from the "anatomy of integers" (elementary/combinatorial number theory).

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4. Arithmetic + Algebraic Geometry:

Bounds on $\mathbb{F}_\ell$-rational points of certain affine varieties over $\overline{\mathbb{F}}_\ell$.

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- Halász's Theorem + estimation of "pretentious distances".
- Modification of the Landau–Selberg–Delange method (mean values of multiplicative functions).

Note: Direct use of mean value estimates is not enough!

3. Character sum machinery + Linear algebra over rings:

Extensions of the Weil bounds + Smith normal forms to bound certain character sums.

4. Arithmetic + Algebraic Geometry:

Bounds on $\mathbb{F}_\ell$-rational points of certain affine varieties over $\overline{\mathbb{F}}_\ell$.

- Lang-Weil bound + study of regular sequences in $\overline{\mathbb{F}}_\ell[X_1, \ldots, X_r]$.

Some of the General Main Results

Consider polynomially-defined multiplicative functions $f_1, \ldots, f_K : \mathbb{Z}^+ \to \mathbb{Z}$, and $q \in \mathbb{Z}^+$.

Narkiewicz (1982): Complete description of the set

$$\mathcal{Q}(f_1, \ldots, f_K) := \{q \in \mathbb{Z}^+ : \ f_1, \ldots, f_K \text{ jointly WUD mod } q\}$$

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Theorem 7 (S.R., 2023-'24).

Under two technical hypotheses H_1 and H_2, the functions $f_1, \ldots, f_K$ are jointly WUD uniformly modulo $q \in \mathcal{Q}(f_1, \ldots, f_K)$ such that $q \le (\log x)^{c_q}$, for some parameter $c_q := c(q; f_1, \ldots, f_K) > 0$.

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Optimality:

1. c_q is optimal in most cases, hence so is the range of q.
2. Optimal in arithmetic restrictions on q.
3. Hypotheses H_1 and H_2 are both necessary.

As for $(\varphi, \sigma, \sigma_2)$, we need to restrict our input sets to get complete uniformity up to arbitrary powers of $\log x$.

Theorem 8 (S.R., 2023-'24).

Fix $K_0 > 0$. Under H_1 and H_2, we have

$$\#\{n \le x : P_R(n) > q, \quad (\forall i)\ f_i(n) \equiv a_i \pmod q\}$$
$$\sim \frac{1}{\varphi(q)^K} \# \left\{ n \le x : P_R(n) > q,\ \gcd(\prod_{i=1}^{K} f_i(n), q) = 1 \right\},$$

uniformly in $q \le (\log x)^{K_0}$ lying in $\mathcal{Q}(f_1, \ldots, f_K)$ and in $a_i \in U_q$.

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Optimality: Most of these R's are either exactly or nearly optimal, ensuring joint WUD among as large a set of inputs as possible.

Thank you for your attention.

A Very Happy Birthday to Prof. Nathanson and Prof. Pomerance!