# The Landau-Selberg-Delange method for Dirichlet $L$-functions, and applications

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**ELAZ 2026**

February 18, 2026

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$$\sum_{n\leq x} a_n = \frac{x}{(\log x)^{1-\alpha}} \sum_{j=0}^{N} \frac{\kappa_j}{(\log x)^j} + O\,(\text{Error Terms}) \qquad (1)$$

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**Extension?** Assume $\sum_n a_n/n^s$ "behaves like" $\prod_{\chi \bmod q} L(s,\chi)^{\alpha_\chi}$, for some $\{\alpha_\chi\}_\chi \subset \mathbb{C}$.

- Give as **precise** asymptotic series estimating $\sum_{n \le x} a_n$, **uniformly** in $q$ in a wide range (wider than from previous literature).

# One of the main results

**Theorem 1 (S.R. 2025, outline).**
*Assume* $\sum_n a_n/n^s$ *"behaves like"* $\prod_{\chi \bmod q} L(s\nu, \chi)^{\alpha_\chi}$*, for some* $\{\alpha_\chi\}_\chi \subset \mathbb{C}$ *and some fixed* $\nu \in \mathbb{R}^+$*. Then for any fixed* $K_0 > 0$*,*

$$\sum_{n \le x} a_n = \frac{x^{1/\nu}}{(\log x)^{1-\alpha_{\chi_0}}} \sum_{0 \le j \le N} \frac{\kappa_j}{(\log x)^j} \ + \ O(\text{Error Terms}),$$

*uniformly in* $x \ge 3$*,* $N \ge 0$ *and* $q \le (\log x)^{K_0}$*.*

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*uniformly in* $x \ge 3$, $N \ge 0$ *and* $q \le (\log x)^{K_0}$.

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Wider ranges under Landau-Siegel zeros conjecture and GRH.

# Some Applications

(1) Positive integers with prime divisors restricted to residue classes:
Given $q \in \mathbb{Z}^+$ and $\mathcal{A} \subset (\mathbb{Z}/q\mathbb{Z})^\times$, estimate
$\#\{n \le x : \ p \mid n \implies p \bmod q \in \mathcal{A}\}$.

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(2) Distributions of the least invariant factor of multiplicative groups:
Writing $(\mathbb{Z}/n\mathbb{Z})^\times \cong \mathbb{Z}/\lambda_1\mathbb{Z} \oplus \mathbb{Z}/\lambda_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\lambda_r\mathbb{Z}$ with $\lambda_i \in \mathbb{Z}^+$, $\lambda_1 \mid \lambda_2 \mid \cdots \mid \lambda_r$, let $\lambda_1(n) := \lambda_1$. Estimate $\#\{n \le x : \lambda_1(n) = d\}$.

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- Chang–Martin (2020): Fixed $d \in \mathbb{Z}^+$.
- Theorem 1 $\implies$ Uniformly in $d \le (\log x)^{K_0}$, much better error terms.

# Further Applications

(3) Residue-class distribution of multiplicative functions: Given multiplicative functions $f_1, \ldots, f_K : \mathbb{Z}^+ \to \mathbb{Z}$, estimate $\#\{n \le x : (\forall i)\, f_i(n) \equiv a_i \pmod q\}$ for units $a_i$ mod $q \le (\log x)^{K_0}$.

- Extends work of Narkiewicz, Rayner, Śliwa, Dobrowolski, Fomenko.
- Essentially optimal analogues of Siegel–Walfisz for large classes of multiplicative functions (and families of such functions).

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(4) Sathe–Selberg Theorem in arithmetic progressions:

- Sathe-Selberg: Local distributions of the functions $\omega(n) = \sum_{p|n} 1$ and $\Omega(n) = \sum_{p|n} v_p(n)$.

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- Sathe-Selberg: Local distributions of the functions $\omega(n) = \sum_{p|n} 1$ and $\Omega(n) = \sum_{p|n} v_p(n)$.
- Theorem 1 $\implies$ Local distributions (modulo $q \le (\log x)^{K_0}$) of

$$\omega_a(n) := \sum_{\substack{p|n \\ p \equiv a \bmod q}} 1 \quad \text{and} \quad \Omega_a(n) := \sum_{\substack{p|n \\ p \equiv a \bmod q}} v_p(n).$$

Thank you for your attention!

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