The Landau-Selberg-Delange method for Dirichlet L-functions, and applications

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$\rightsquigarrow$ Tauberian theory, mean values of multiplicative functions.

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6. Distribn. of invariant factors and elementary divisors of unit groups.

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Let χ be a Dirichlet character mod q. $L(s,\chi) := \sum_{n\geq 1} \chi(n)/n^s$.

$\chi : \mathbb{Z} \to \mathbb{C}$ is periodic with period q and satisfies

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Behavior of $F(s)$ on $\{\mathrm{Re}(s) = b\}$? All bets are off!

Eg: $\sum_{n\geq 1} \frac{1}{n}$ diverges, but $\sum_{n\geq 2} \frac{(-1)^n \cdot n/\log^2 n}{n^s}$ converges on $\mathrm{Re}(s) = 1$.

Tauberian Theorems

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(i) $F(s)$ converges absolutely on $\{\mathrm{Re}(s) > 1\}$, and

(ii) There exist $\alpha > 0$ and a function $G(s)$ holomorphic in a neighborhood of $s = 1$ with $G(1) \ne 0$, s. t. $F(s) = G(s)/(s-1)^\alpha$ on this neighborhood.

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Then $$\sum_{n \le x} a_n \sim \frac{c_0\, x}{(\log x)^{1-\alpha}} \text{ as } x \to \infty,$$ for some constant $c_0 \in \mathbb{C}$.

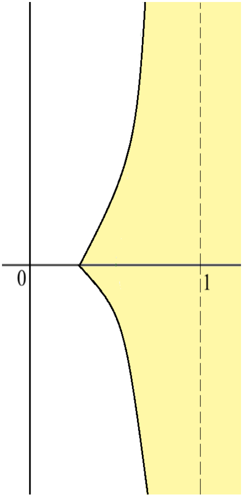
The Classical Landau-Selberg-Delange "LSD" Setting

Basic set-up: Often it happens that

$$\sum_{n\geq 1} \frac{a_n}{n^s} = \zeta(s)^\alpha G(s) \text{ for all } s \text{ with } \mathrm{Re}(s) > 1,$$

where $\alpha \in \mathbb{C}$, and $G(s)$ is "well-behaved".

$G(s)$: analytic & bounded as a "nice" function of s, on a region like...



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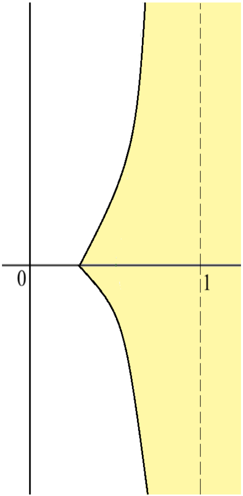

Why this region? $\zeta(s)$ can be continued meromorphically into a nonvanishing function on this region.

- Only simple pole at $s = 1$.

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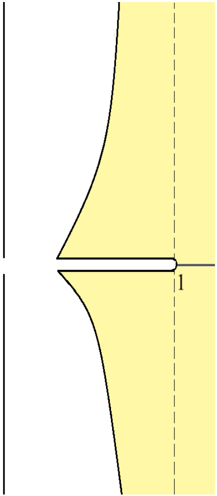
- $\zeta(s)^\alpha$ can be analytically continued into a region like the one shown.

  Note: Possible branch point at $s = 1$.



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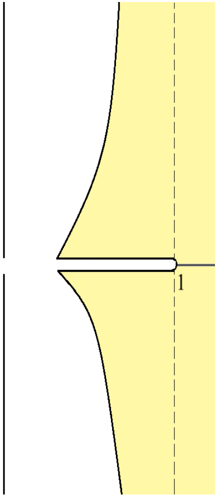
Intuition: Why this assumption?

- When $\{a_n\}_n$ is **multiplicative** (i.e. $a_{mn} = a_m\, a_n$ for $\gcd(m,n) = 1$), problems on $\{a_n\}_n$ reduce to $\{a_p\}_p$.



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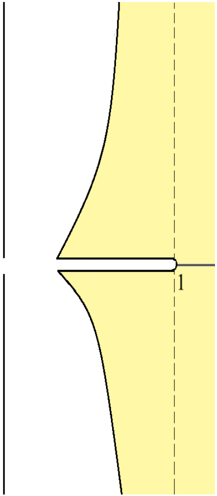

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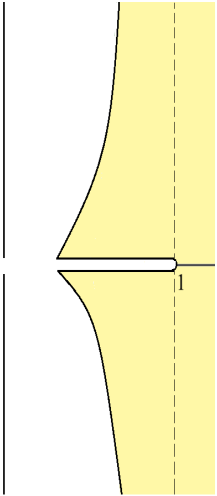
- In this case, often $\alpha \approx \sum_p a_p / \sum_p 1$.

Eg: $a_n = \mathbb{1}(p \mid n \implies p \equiv 1 \bmod 4)$
$\implies a_p = \mathbb{1}_{p \equiv 1 \bmod 4} \implies \alpha = 1/2$.



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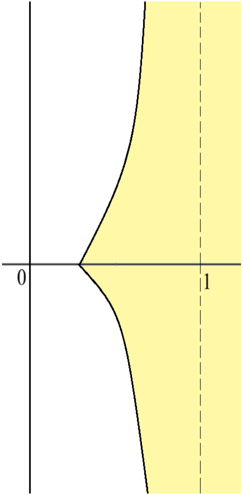

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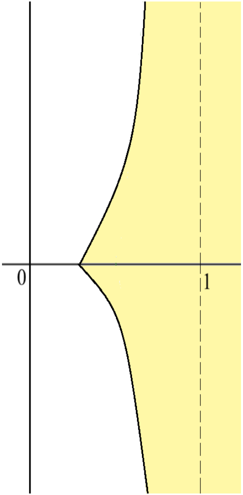

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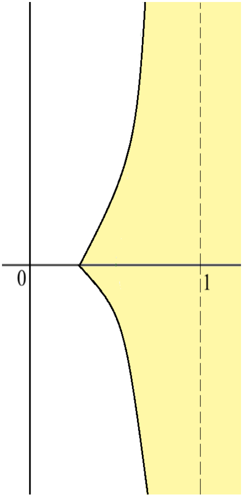
Delange-Ikehara $\Longrightarrow \displaystyle\sum_{n\leq x} a_n \sim \frac{c_0\, x}{(\log x)^{1-\alpha}}$.



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Theorem ("The LSD Method", Tenenbaum). Assume that $\sum_{n\geq 1} a_n/n^s = \zeta(s)^\alpha G(s)$ for all s with $\mathrm{Re}(s) > 1$, where $\alpha \in \mathbb{C}$, and $G(s)$ is "well-behaved". Then for some $c_0, \ldots, c_N \in \mathbb{C}$,

$$\sum_{n\leq x} a_n = \frac{c_0\, x}{(\log x)^{1-\alpha}} + \frac{c_1\, x}{(\log x)^{2-\alpha}} + \cdots + \frac{c_N\, x}{(\log x)^{N+1-\alpha}} + O(\mathrm{Err}),$$

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- **Err:** Depends on how "well-behaved" G is.
  - Does **NOT** always give the desired saving!

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$$\sum_{n\ge1} \frac{\mathbb{1}_{n=\square+\square}}{n^s} = \sum_{m,a,b} \frac{1}{2^{ms}\, a^{2s}\, b^s} = \left(\sum_{m\ge0} \frac{1}{2^{ms}}\right)\left(\sum_{a} \frac{1}{a^{2s}}\right)\left(\sum_{b} \frac{1}{b^s}\right)$$

$$= \left(1 - \frac{1}{2^s}\right)^{-1} \cdot \prod_{\ell\equiv 3 \bmod 4} \left(1 + \frac{1}{\ell^{2s}} + \frac{1}{\ell^{4s}} + \dots\right)$$

$$\cdot \prod_{p\equiv 1 \bmod 4} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right)$$

Typical term of last product $= 1/(p_1^{e_1} \cdots p_k^{e_k})^s$ with $p_1 < \cdots < p_k$ primes $\equiv 1 \bmod 4$, and $e_1, \dots, e_k \in \mathbb{Z}^+$. Every $1/b^s$ appears exactly once.

$$\sum_{n\geq 1} \frac{\mathbb{1}_{n=\square+\square}}{n^s}$$

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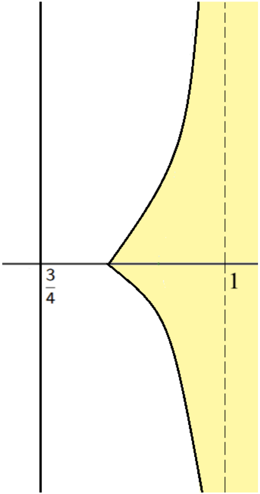

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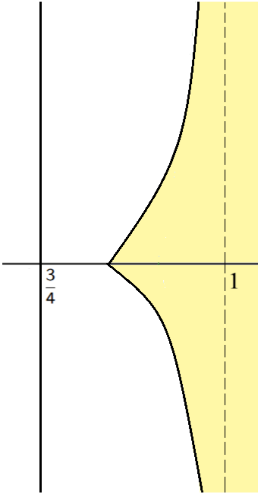

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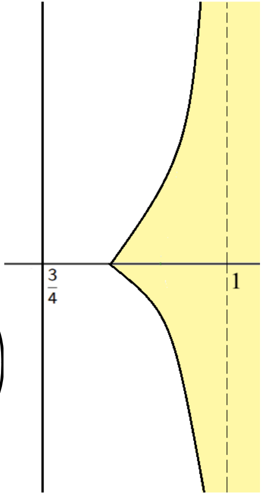

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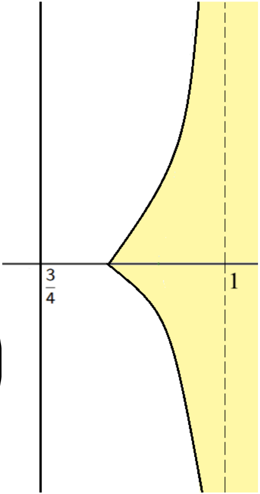

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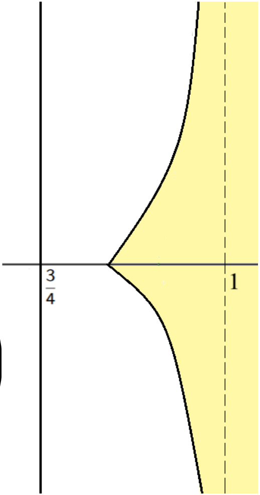

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Extending the LSD theorem...

Tenenbaum: Assume $\sum_{n\geq 1} a_n/n^s = \zeta(s)^\alpha G(s)$ for $\mathrm{Re}(s) > 1$, where $\alpha \in \mathbb{C}$, and $G(s)$ is well-behaved. Then uniformly in $x \geq 3$, $N \geq 0$,

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Eg.: Let $a_n = \mathbb{1}(p \mid n \implies p \equiv a \bmod q)$. Then $\alpha_\chi = \overline{\chi}(a)/\varphi(q)$.

Not as simple as it might seem...

Facts

(i) With $\chi_0(n) := \mathbb{1}_{(n,q)=1}$, we have $L(s, \chi_0) \approx \zeta(s)$ simple pole at $s = 1$.

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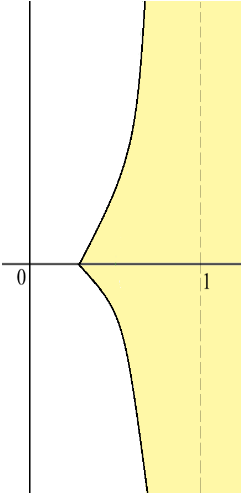

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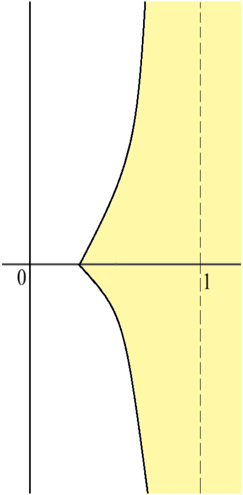
$= \left(\prod_{\chi \bmod q} L(s,\chi)^{\alpha_\chi}\right) \cdot G(s) \approx \zeta(s)^{\alpha_{\chi_0}} \cdot H(s)$?



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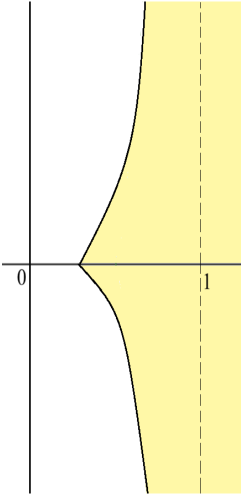
needs to be well-behaved.

analytic + suitably bounded as a function of s in a region like...



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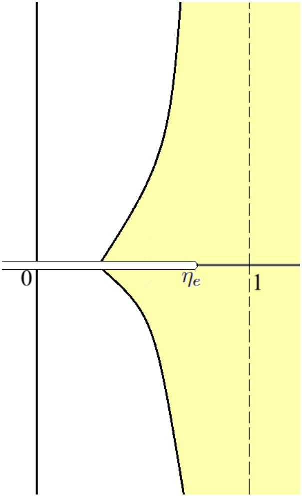

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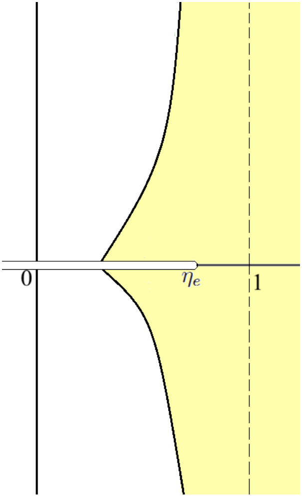
Issue 2: Best known direct bounds on $H(s)$ often grow far too rapidly with q.

$\Longrightarrow$ SEVERELY impede uniformity in q.



One of the main results...

Theorem 1 (S.R. '25). Fix $K_0 > 0$. Assume for s with $\mathrm{Re}(s) > 1$, that $\sum_{n\geq 1} a_n/n^s = \left(\prod_{\chi \bmod q} L(s,\chi)^{\alpha_\chi}\right) \cdot G(s)$, where $\{\alpha_\chi\}_\chi \subset \mathbb{C}$, and $G(s)$ is well-behaved. Then for some $\{c_j\}_{j\geq 1} \subset \mathbb{C}$, we have **uniformly** in $x \geq 3$, $N \geq 0$ and $q \leq (\log x)^{K_0}$,

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- Conditionally on Generalized Riemann Hypothesis/Landau–Siegel zeros conjecture $\implies$ Wider uniformity ranges in q.

Appl. 1. Integers supported on primes in progressions

Problem: Given $q \in \mathbb{Z}^+$ and $\mathcal{A} \subset (\mathbb{Z}/q\mathbb{Z})^\times$, estimate

$$\mathcal{N}(x; q, \mathcal{A}) := \#\{n \leq x : p \mid n \Longrightarrow p \bmod q \in \mathcal{A}\}.$$

- **Landau (1908):** $\mathcal{N}(x; 4, \{1 \bmod 4\})$ (distribution of $n = \square + \square$).

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✓ NOTE (for all applications): Better ranges on q conditionally.

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$(\mathbb{Z}/n\mathbb{Z})^\times \cong \mathbb{Z}/\lambda_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\lambda_r\mathbb{Z}$ with $\lambda_i \in \mathbb{Z}^+$ s.t. $\lambda_1 \mid \ldots \mid \lambda_r$.

Least invariant factor: $\boldsymbol{\lambda^*(n) := \lambda_1}$.

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Theorem 1 $\Longrightarrow$ distribution of **least elementary divisor** of $(\mathbb{Z}/n\mathbb{Z})^\times$
$\rightsquigarrow$ extending work of **Martin–Nguyen (2024)**.

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- **Rankin, Scourfield, Serre, Spearman–Williams, Narkiewicz, Ford–Luca–Moree, . . . :** specific interesting f and fixed q.

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Problem: Given multiplicative $f : \mathbb{Z}^+ \to \mathbb{Z}$ and $q \in \mathbb{Z}^+$, estimate

$$\#\{n \le x : \gcd(f(n), q) = 1\}.$$

- **Rankin, Scourfield, Serre, Spearman–Williams, Narkiewicz, Ford–Luca–Moree, . . . :** specific interesting f and fixed q.
- **Theorem 1** $\Longrightarrow$ Extend to $q \le (\log x)^{K_0}$ and more general f.

Appl. 4. Sathe-Selberg in arithmetic progressions

$\omega(n) := \sum_{p|n} 1 := \#\{p \mid n\}, \qquad \Omega(n) := \sum_{p|n} v_p(n).$

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$$\omega_a(n) := \sum_{p|n:\ p\equiv a \pmod q} 1, \qquad \Omega_a(n) := \sum_{p|n:\ p\equiv a \pmod q} v_p(n) \quad ?$$

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Theorem 1 $\Longrightarrow$ estimate $\#\{n \le x : f(n) = k\}$ for $f \in \{\omega_a, \Omega_a\}$, uniformly in $q \le (\log x)^{K_0}$ and in $a \in (\mathbb{Z}/q\mathbb{Z})^\times$.

Appl. 5. Residue-class distribution of multiplicative functions

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General: Given multiplicative $f_1, \ldots, f_K : \mathbb{Z}^+ \to \mathbb{Z}$ and $q \in \mathbb{Z}^+$, $a_i \in (\mathbb{Z}/q\mathbb{Z})^\times$, estimate $\#\{n \le x : (\forall i)\ f_i(n) \equiv a_i \pmod{q}\}$.

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Aside (Ingredients): "Mixing" in $(\mathbb{Z}/q\mathbb{Z})^\times$, detected via anatomy of integers, character sum machinery, linear algebra over rings, arithmetic geometry, algebraic geometry, Halász + Theorem 1.

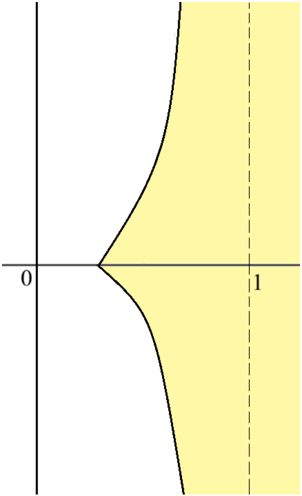
Summary of the main ideas

Set-up: $\sum_{n\geq 1} a_n/n^s = \mathcal{F}(s)G(s)$, with

$$\mathcal{F}(s) := \prod_{\chi \bmod q} L(s,\chi)^{\alpha_\chi},$$

and $G(s)$ well-behaved.

To estimate: $$\sum_{n\leq x} a_n.$$



Main ideas behind Theorem 1: Summarized

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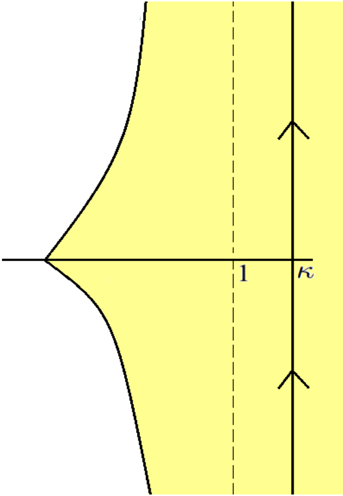
Step 1. Perron's formula

$$\sum_{n\leq x} a_n = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{\mathcal{F}(s)G(s)x^s}{s}\, ds.$$



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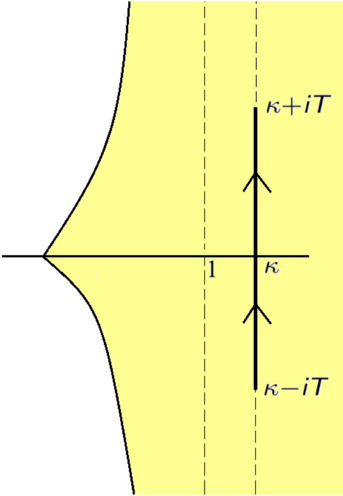

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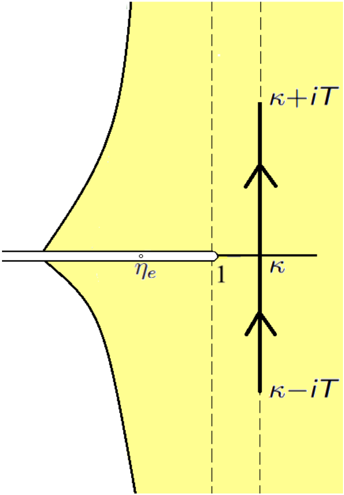
Step 2. $\mathcal{F}(s)$ analytically continues into the shaded region.

- Two possible branch points, viz. $s=1$ and $s=\eta_e$.



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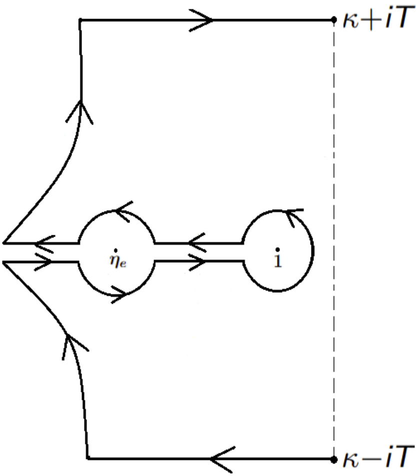

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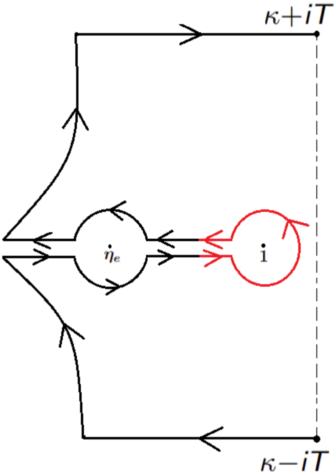
Step 3. Red part → "Hankel contour"
$\implies$ main term, secondary term, ...



Bounding the contribution of the rest of Γ

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- For suitable parameters $\xi := \xi(q;s)$ and $B := B(s)$, consider

$$\mathcal{G}(s) \quad := \quad \int_{z=B-i\infty}^{z=B+i\infty} \frac{\xi^{z-s} - \xi^{2(z-s)}}{(z-s)^2} \cdot \frac{\mathcal{F}'(z)}{\mathcal{F}(z)} \, \mathrm{d}z.$$

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$\longrightarrow$ Split into regions + zero-density estimates.

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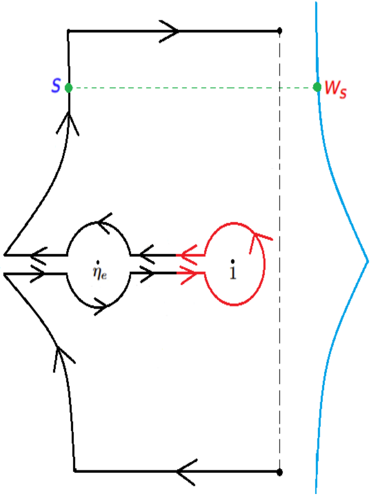

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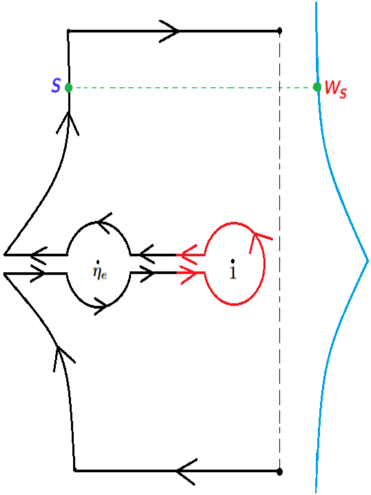

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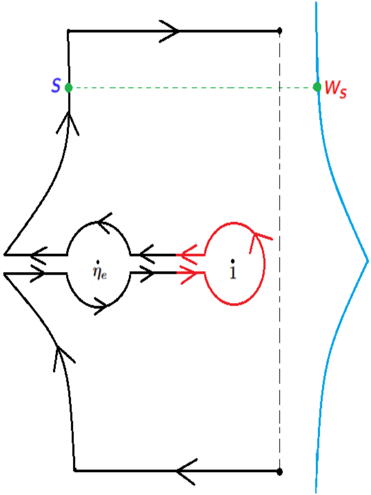

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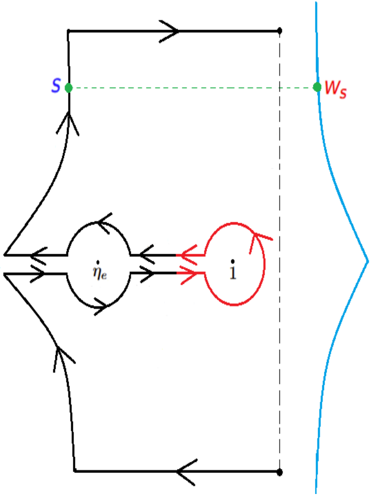
General Case: $\sum_{n\geq 1} a_n/n^s$
$$= \left(\prod_{\chi \bmod q} L(s\nu, \chi)^{\alpha_\chi}\right) G(s)$$

$\rightsquigarrow$ scaling + averaging + "pigeonhole".



Thank you for your attention!

Email: akash01s.roy@gmail.com