The Landau-Selberg-Delange method for Dirichlet L-functions, and applications

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General problem: Given a Dirichlet series $\sum_n a_n/n^s$ with natural analytic properties, give suitable estimates on $\sum_{n\leq x} a_n$.

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$$\sum_{n\le x} a_n = \frac{x}{(\log x)^{1-\alpha}} \sum_{j=0}^{N} \frac{\kappa_j}{(\log x)^j} + O\,(\text{Error Terms}) \qquad (1)$$

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Extension? Assume $\sum_n a_n/n^s$ "behaves like" $\prod_{\chi \bmod q} L(s,\chi)^{\alpha_\chi}$, for some $\{\alpha_\chi\}_\chi \subset \mathbb{C}$.

- Give as **precise** asymptotic series estimating $\sum_{n\le x} a_n$, **uniformly** in q in a wide range (wider than from previous literature).

One of the main results

Theorem 1 (S.R. 2025, outline).
Assume $\sum_n a_n/n^s$ "behaves like" $\prod_{\chi \bmod q} L(s\nu,\chi)^{\alpha_\chi}$, for some $\{\alpha_\chi\}_\chi \subset \mathbb{C}$ and some fixed $\nu \in \mathbb{R}^+$. Then for any fixed $K_0 > 0$,

$$\sum_{n \le x} a_n = \frac{x^{1/\nu}}{(\log x)^{1-\alpha_{\chi_0}}} \sum_{0 \le j \le N} \frac{\kappa_j}{(\log x)^j} \; + \; O(\text{Error Terms}),$$

uniformly in $x \ge 3$, $N \ge 0$ and $q \le (\log x)^{K_0}$.

Error term: Genuine saving over main term in the <u>full range</u> $q \le (\log x)^{K_0}$ in several applications of interest.

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Wider ranges under Landau-Siegel zeros conjecture and GRH.

Some applications

(1) Positive integers with prime divisors restricted to residue classes:
Given $q \in \mathbb{Z}^+$ and $\mathcal{A} \subset (\mathbb{Z}/q\mathbb{Z})^\times$, estimate
$\#\{n \le x : \ p \mid n \implies p \bmod q \in \mathcal{A}\}$.

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- Sathe-Selberg: Local distributions of the functions $\omega(n) = \sum_{p|n} 1$ and $\Omega(n) = \sum_{p|n} v_p(n)$.

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$$\omega_a(n) := \sum_{\substack{p|n \\ p\equiv a \bmod q}} 1 \quad \text{and} \quad \Omega_a(n) := \sum_{\substack{p|n \\ p\equiv a \bmod q}} v_p(n).$$

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- Theorem 1 $\implies$ Local distributions of $\omega_a(n)$ and $\Omega_a(n)$ modulo $q \le (\log x)^{K_0}$.

One more application for now

(3) Residue-class distribution of multiplicative functions: Given multiplicative functions $f_1, \dots, f_K : \mathbb{Z}^+ \to \mathbb{Z}$, estimate $\#\{n \le x : (\forall i)\, f_i(n) \equiv a_i \pmod q\}$ for units a_i mod q.

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  ✓Large classes of $f_1, \ldots, f_K$ and **fixed** q.
- Theorem 1 $\Longrightarrow$ ✓ Extend to $q \le (\log x)^{K_0}$
  ✓ Qualitatively + Quantitatively optimal results.

Thank you for your attention!

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